STOKES CONSTANTS IN TOPOLOGICAL STRING THEORY

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MOTIVATIONS AND BACKGROUND

Enumerative invariants and resurgence

Asymptotic, resurgent series arise naturally as perturbative expansions in quantum theories.

The machinery of resurgence uniquely associates to them a collection of complex numbers, known as **Stokes constants**, which capture information about the non-perturbative sectors of the theory.

In some remarkable cases, the Stokes constants can be interpreted in terms of **enumerative invariants** based on the counting of BPS states.

Some recent developments:

- 4d $\mathcal{N} = 2$ supersymmetric gauge theory in the Nekrasov-Shatashvili limit of the Omega-background

[Grassi, Gu, Mariño, 2020]

- Complex Chern-Simons theory on the complement of a hyperbolic knot

[Garoufalidis, Gu, Mariño, 2020 - 2022]

- Standard topological string theory on a Calabi-Yau threefold in the weakly-coupled regime

[Gu, Mariño, 2022]

FROM TOPOLOGICAL STRINGS TO QUANTUM OPERATORS AND BACK

From topological strings to quantum operators

Let X be a toric Calabi-Yau (CY) threefold.

Local mirror symmetry pairs X with an algebraic curve Σ of genus g_Σ, whose quantization leads to **quantum-mechanical operators**

$$\rho_j\,, \quad j = 1, \ldots, g_\Sigma\,,$$

acting on $L^2(\mathbb{R})$, and conjectured to be positive-definite and of trace class, under some assumptions on the mass parameters ξ.
[Grassi, Hatsuda, Mariño, 2016 - Codesido, Grassi, Mariño, 2017]

Their **Fredholm determinant** $\Xi(\kappa, \xi, \hbar)$ is an entire function of the true complex moduli κ, and its local expansion at the orbifold point $\kappa_j = 0$, that is,

$$\Xi(\kappa, \xi, \hbar) = \sum_{N_1 \geq 0} \cdots \sum_{N_{g_\Sigma} \geq 0} Z(\mathbf{N}, \xi, \hbar)\, \kappa_1^{N_1} \cdots \kappa_{g_\Sigma}^{N_{g_\Sigma}}\,,$$

defines the **fermionic spectral traces** $Z(\mathbf{N}, \xi, \hbar)$, which are analytic functions of $\hbar \in \mathbb{R}_{>0}$.

We will assume that $Z(\mathbf{N}, \xi, \hbar)$ can be analytically continued to $\hbar \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

From quantum operators to topological strings

The **Topological Strings/Spectral Theory** (TS/ST) correspondence gives

$$Z(\mathbf{N},\xi,\hbar) = \frac{1}{(2\pi \mathrm{i})^{g_\Sigma}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu_1 \cdots \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu_{g_\Sigma} \exp(J(\mu,\xi,\hbar) - \mathbf{N}\cdot\mu)\,,$$

where the chemical potentials μ_j are defined by $\kappa_j = e^{\mu_j}$.
[Hatsuda, Moriyama, Okuyama, 2013 - Grassi, Hatsuda, Mariño, 2016 - Codesido, Grassi, Mariño, 2017]

The **total grand potential** $J(\mu,\xi,\hbar)$ can be written as

$$J(\mu,\xi,\hbar) = J^{\mathrm{WS}}(\mu,\xi,\hbar) + J^{\mathrm{WKB}}(\mu,\xi,\hbar)\,,$$

where the worldsheet instanton grand potential J^{WS} and the WKB grand potential J^{WKB} capture the contributions from the standard and the Nekrasov-Shatashvili (NS) topological strings, respectively.
[Hatsuda, Mariño, Moriyama, Okuyama, 2014]

The standard topological string coupling constant g_s is related to the quantum deformation parameter $\hbar$ by $g_s = 4\pi^2/\hbar$ (**strong-weak coupling duality**).

NOTIONS FROM THE THEORY OF RESURGENCE

Resurgence in quantum theories — I

Let $\phi(z)$ be a resurgent Gevrey-1 asymptotic series of the form

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[\![z]\!]\,, \quad a_n \sim A^{-n} n! \quad n \gg 1\,.$$

Its **Borel resummation** is the two-step process

$$\phi(z) \rightsquigarrow \hat{\phi}(\zeta) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \zeta^k \rightsquigarrow s_\theta(\phi)(z) = \int_{\rho_\theta} e^{-\zeta} \hat{\phi}(\zeta z)\, d\zeta\,,$$

where $\rho_\theta = e^{i\theta}\mathbb{R}_+$, $\theta = \arg(\zeta)$.

The Borel resummation $s_\theta(\phi)(z)$ is a locally analytic function in the complex z-plane with discontinuities at

$$\arg(z) = \arg(\zeta_\omega)\,,$$

where ω labels the singularities $\zeta_\omega \in \mathbb{C}$ of $\hat{\phi}(\zeta)$.

A ray containing one or more singularities of $\hat{\phi}(\zeta)$ is called a **Stokes ray**.

The **discontinuity** of the Borel resummation across the Stokes ray ρ_θ is given by

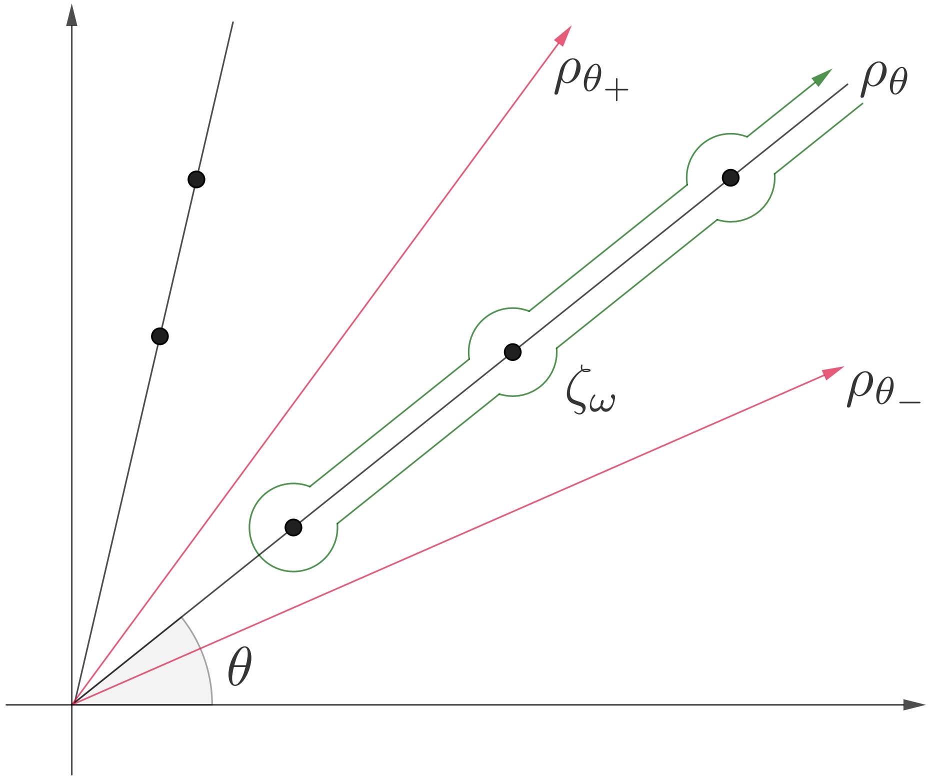
$$\mathrm{disc}_\theta \phi(z) = s_{\theta_+}(\phi)(z) - s_{\theta_-}(\phi)(z) = \sum_\omega S_\omega \, e^{-\zeta_\omega/z} \, s_{\theta_-}(\phi_\omega)(z) \,,$$

where $\theta_\pm = \theta \pm \epsilon$ for some small positive angle ϵ, and the sum is performed over all the singularities ζ_ω of $\hat{\phi}(\zeta)$ which lie on ρ_θ.

The **Stokes constant** $S_\omega \in \mathbb{C}$ and the resurgent Gevrey-1 asymptotic series $\phi_\omega(z)$ are encoded in the local expansion of the Borel transform at the singularity $\zeta = \zeta_\omega$. If ζ_ω is a logarithmic branch point, we have

$$\hat{\phi}(\zeta) = -\frac{S_\omega}{2\pi i} \log(\zeta - \zeta_\omega) \hat{\phi}_\omega(\zeta - \zeta_\omega) + \ldots \,,$$

where the dots denote regular terms in $\xi = \zeta - \zeta_\omega$, and $\hat{\phi}_\omega(\xi)$ is the Borel transform of $\phi_\omega(z)$.



The **Stokes automorphism** $\mathfrak{S}_\theta$ across ρ_θ is defined by $s_{\theta_+} = s_{\theta_-} \circ \mathfrak{S}_\theta$.

Resurgence in quantum theories — III

We can repeat the procedure with each of the series obtained in this way. Schematically,

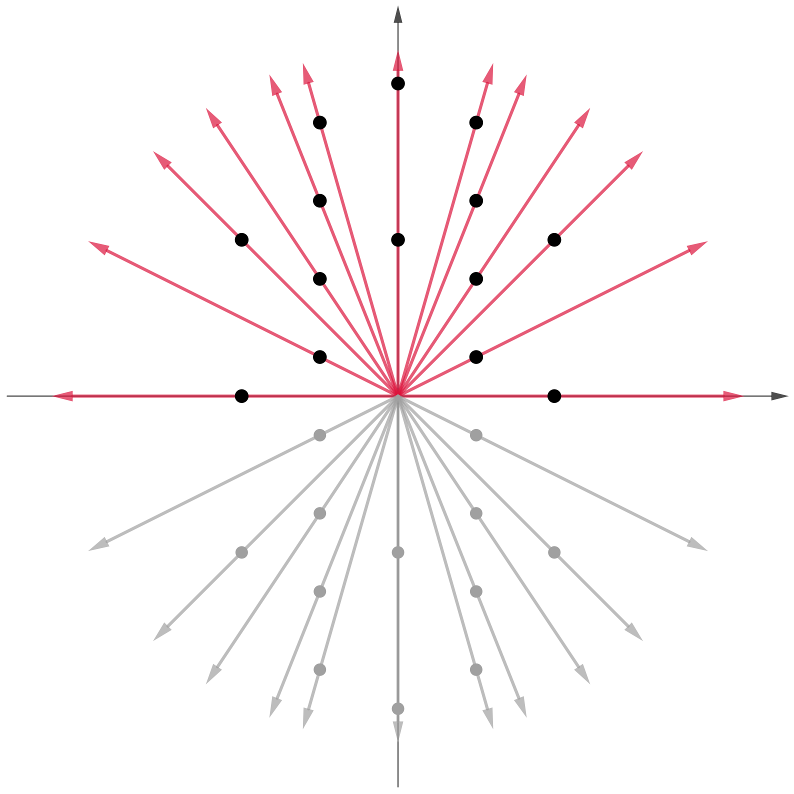
$$\phi \rightsquigarrow \{\phi_\omega, S_\omega\} \rightsquigarrow \{\phi_{\omega'}, S_{\omega\omega'}\}\,.$$

Each resurgent series in this process can be promoted to **basic trans-series** as

$$\Phi_\omega(z) = e^{-\zeta_\omega/z}\,\phi_\omega(z)\,.$$

The **minimal resurgent structure** associated to $\phi(z)$ is defined as the smallest subset of its basic trans-series which forms a closed system under Stokes automorphisms. We denote it by

$$\mathfrak{B}_\phi = \left\{\Phi_\omega(z)\right\}_{\omega\in\Omega}\,.$$



Evidence suggests that a **peacock pattern** of singularities in the complex Borel plane is typical of theories controlled by a quantum curve in exponentiated variables.

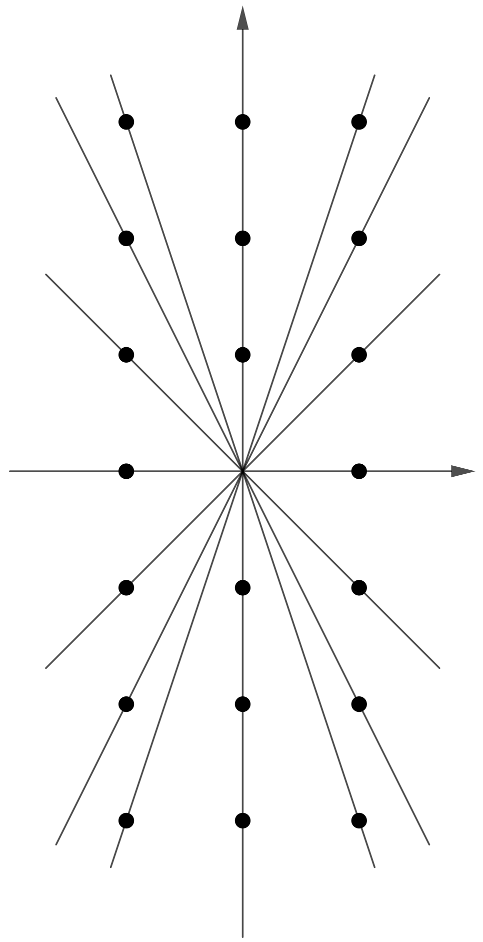
TOPOLOGICAL STRINGS BEYOND PERTURBATION THEORY

[Rella, 2022]

In the semiclassical limit $\hbar \to 0$, the perturbative expansion of $\log Z(\mathbf{N}, \hbar)$ produces a **family of asymptotic series** $\phi_{\mathbf{N}}(\hbar)$, indexed by $\mathbf{N} \in \mathbb{N}^{g_\Sigma}$, which I assume to be Gevrey-1 and resurgent. I denote

$$\psi_{\mathbf{N}}(\hbar) = \exp(\phi_{\mathbf{N}}(\hbar)) \,.$$

A **periodic peacock configuration** of Stokes rays is expected to occur in the complex Borel plane of the perturbative series $\phi_{\mathbf{N}}(\hbar)$ for fixed $\mathbf{N}$.



More precisely, there is a finite number of Gevrey-1 asymptotic series

$$\phi_{\sigma;\mathbf{N}}(\hbar) \,, \quad \sigma \in \{0, \ldots, l\} \,,$$

which resurge from $\phi_{\mathbf{N}}(\hbar) = \phi_{0;\mathbf{N}}(\hbar)$.

For each σ, there is an infinite family of basic trans-series

$$\Phi_{\sigma,n;\mathbf{N}}(\hbar) = \phi_{\sigma;\mathbf{N}}(\hbar)\, e^{-n\frac{A}{\hbar}} \,, \quad n \in \mathbb{N} \,,$$

where $A \in \mathbb{C}$ and $l \in \mathbb{N}_+$ depend on $\mathbf{N}$ and on the CY geometry.

Stokes constants and peacock patterns in topological string theory — II

The basic trans-series $\Phi_{\sigma,n;\mathbf{N}}(\hbar)$ capture explicit non-analytic corrections in $\hbar$, which describe **additional, hidden sectors** of the topological string.

They lead to a minimal resurgent structure of the form

$$\mathfrak{B}_{\phi_{\mathbf{N}}} = \{\Phi_{\sigma,n;\mathbf{N}}(\hbar)\}_{\sigma=0,\ldots,l,\, n\in\mathbb{N}}\,,$$

and to **infinitely-many Stokes constants**

$$S_{\sigma\sigma',n;\mathbf{N}}\,, \quad \sigma,\sigma' = 0,\ldots,l\,, \quad n\in\mathbb{N}\,,$$

which I conjecture to be rational numbers.

In the **dual weakly-coupled limit** $g_s \propto \hbar^{-1} \to 0$, the perturbative expansion of $\log Z(\mathbf{N},\hbar)$ produces a family of factorially divergent formal power series $\phi_{\mathbf{N}}(g_s)$.

The asymptotic series $\psi_{\mathbf{N}}(g_s) = \exp(\phi_{\mathbf{N}}(g_s))$ are conjectured to give peacock patterns of singularities in their Borel plane, and to infinite sets of **integer Stokes constants**.
[Gu, Mariño, 2022]

Strong and weak coupling limits

I have applied to the strong coupling limit $g_s \to \infty$ the same **resurgent machine** advocated in [Gu, Mariño, 2022] for the weak coupling limit $g_s \to 0$. In summary,

$$\phi_{\mathbf{N}} = \log \psi_{\mathbf{N}} \rightsquigarrow \mathfrak{B}_{\phi_{\mathbf{N}}} = \{\Phi_{\sigma,n;\mathbf{N}}(\hbar)\}_{\sigma=0,\ldots,l,\, n\in\mathbb{N}} \rightsquigarrow \{S_{\sigma\sigma',n;\mathbf{N}}\}_{\sigma,\sigma'=0,\ldots,l,\, n\in\mathbb{N}} .$$

Analogies with $g_s \to 0$:

- The Stokes constants of $\phi_{\mathbf{N}}$ are rational numbers and simply related to non-trivial integer sequences. Their interpretation in terms of **enumerative invariants** of the underlying theory is still missing.
- These Stokes constants are uniquely determined by the original perturbative expansion, and yet they are intrinsically **non-perturbative**.
- Computations in concrete cases resort to the **TS/ST correspondence**.

Differences with $g_s \to 0$:

- The Stokes constants of $\psi_{\mathbf{N}}$ are generally complex numbers, not necessarily **integers**.
- The asymptotic series $\psi_{\sigma;\mathbf{N}}$ do not have an **exponential pre-factor** of the form $e^{-1/\hbar}$, suggesting that there is no direct analogue of the conifold volume conjecture for toric CYs.

THE EXAMPLE OF LOCAL $\mathbb{P}^2$

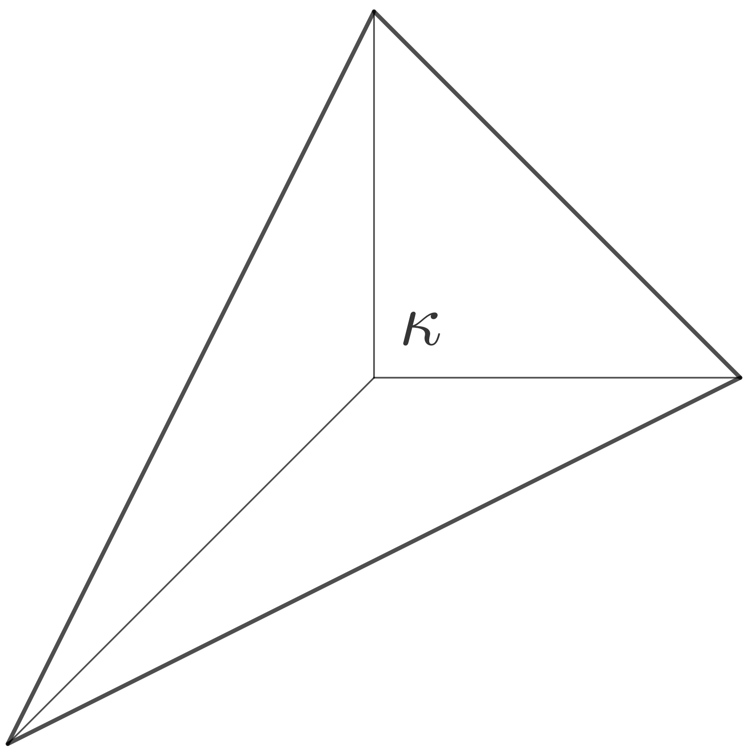
[Rella, 2022]

An introduction to the local $\mathbb{P}^2$ geometry

The simplest example of a **toric del Pezzo CY threefold** is the total space of the canonical bundle over the projective surface $\mathbb{P}^2$, that is,

$$X = \mathscr{O}(-3) \to \mathbb{P}^2 ,$$

called local $\mathbb{P}^2$.



There are one complex modulus κ and no mass parameters.

The first fermionic spectral trace has the **matrix integral representation**

$$Z_{\mathbb{P}^2}(1, \hbar) = \int_{\mathbb{R}} \frac{dy}{2\sqrt{3}\pi b^2} \exp\left(\frac{y}{3} + \log\left(\frac{\Phi_b\left(\frac{y}{2\pi b} + \mathrm{i}\frac{b}{3}\right)}{\Phi_b\left(\frac{y}{2\pi b} - \mathrm{i}\frac{b}{3}\right)} \right) \right) ,$$

where $2\pi b^2 = 3\hbar$, and Φ_b is **Faddeev's quantum dilogarithm**.

[Kashaev, Mariño, 2015]

The first resurgent structure: a numerical approach — I

Expanding in the limit $b \to 0$, and integrating order-by-order in b, I compute numerically the **perturbative series** to very high order.

$$\psi(\hbar) = \frac{\Gamma(1/3)^3}{6\pi\hbar}\left(1 - \frac{\hbar^2}{72} + \frac{23\hbar^4}{51840} - \frac{491\hbar^6}{11197440} + \mathrm{O}(\hbar^8)\right)$$

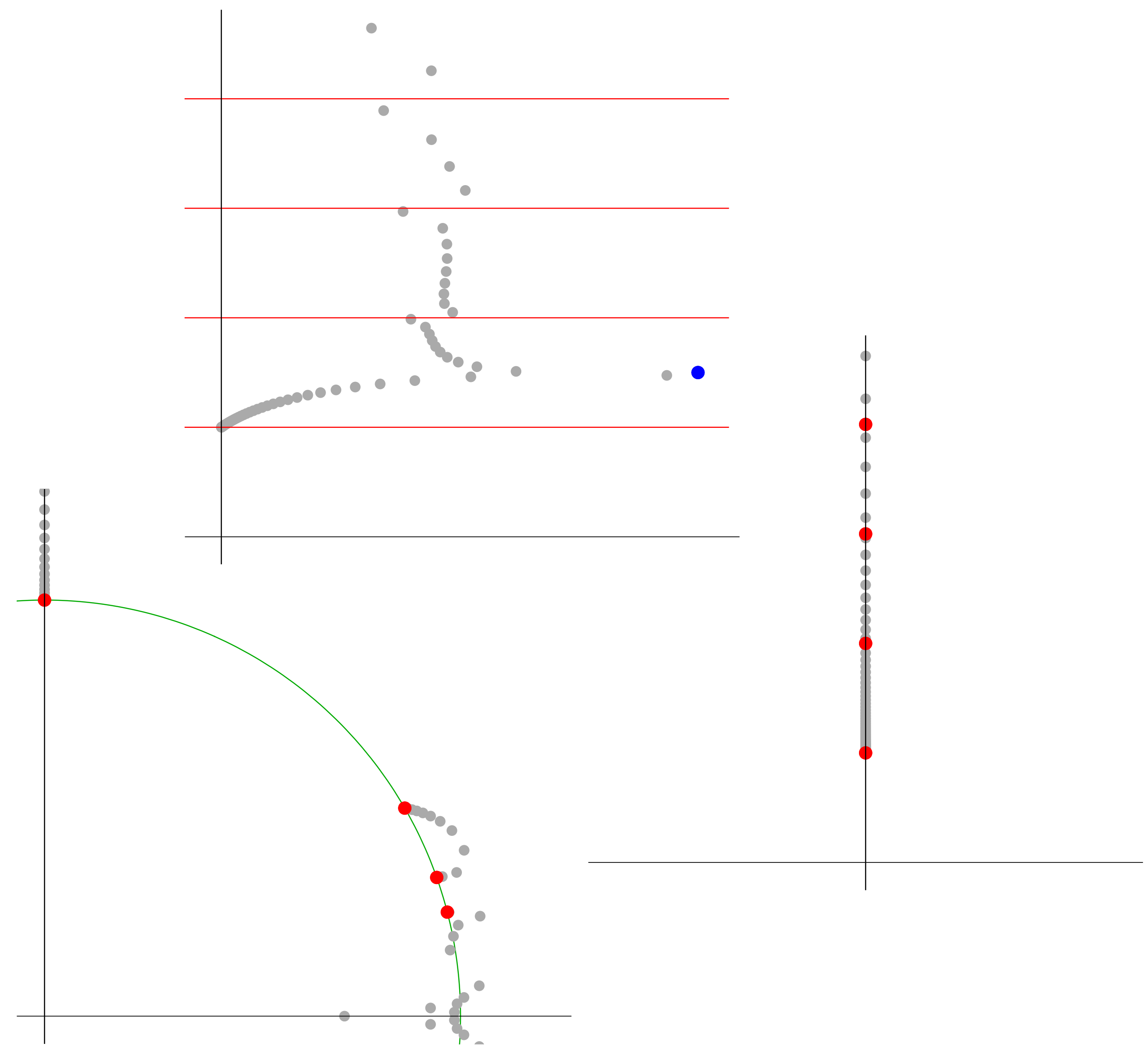
Its coefficients show a **factorial growth** with alternating sign. Namely,

$$b_{2n} \sim (-1)^n \frac{(2n)!}{A^{2n}} \quad n \gg 1\,, \quad A = \frac{4\pi^2}{3}\,.$$

I perform a full **Padé-Borel analysis** with the additional help of conformal maps.
[Costin, Dunne, 2019 - 2020 - 2021]

I find complex conjugate branch points at

$$\zeta = mA\mathrm{i}\,, \quad m \in \mathbb{Z}_{\neq 0}\,.$$



The first resurgent structure: a numerical approach — II

I consider the standard functional ansatz

$$\psi_1(\hbar) = \hbar^{-b} \sum_{k=0}^{\infty} c_k \hbar^k \,,$$

and I test the standard **large-order asymptotics** of the perturbative coefficients

$$n_{2n} \sim \frac{(-1)^n \bar{S}_1}{\pi \mathrm{i}} \frac{\Gamma(2n+b)}{A^{2n+b}} \sum_{k=0}^{\infty} \frac{c_k A^k}{\prod_{j=1}^{k} (2n+b-j)} \qquad n \gg 1 \,.$$

With the help of high-order **Richardson transforms,** I find that

$$A \approx \frac{4\pi^2}{3} \,, \quad b \approx 1 \,, \quad \bar{S}_1 = 3\sqrt{3}\mathrm{i} \,,$$

while the coefficients satisfy

$$c_{2k} \approx b_{2k} \,, \quad c_{2k+1} \approx 0 \,, \quad k \in \mathbb{N} \,,$$

and therefore $\psi_1(\hbar) = \psi(\hbar)$.

The first resurgent structure: an analytic approach — I

In the case of local $\mathbb{P}^2$, the first fermionic spectral trace is known explicitly as
[Kashaev, Mariño, 2015]

$$Z_{\mathbb{P}^2}(1, \hbar \to 0) = \frac{\Gamma(1/3)^3}{6\pi\hbar} \exp\left(3 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_{2n} B_{2n+1}(2/3)}{2n(2n+1)!} (3\hbar)^{2n} \right) .$$

Let $\phi(\hbar)$ be the series in the exponent above.

By application of **Hadamard's multiplication theorem**, I find an explicit formula for the Borel transform $\hat{\phi}(\zeta)$ as an **exact function** of ζ .

This proves that the singularities of $\hat{\phi}(\zeta)$ are **logarithmic branch points** at

$$\zeta = \zeta_m = m \frac{4\pi^2 \mathrm{i}}{3} , \quad m \in \mathbb{Z}_{\neq 0} .$$

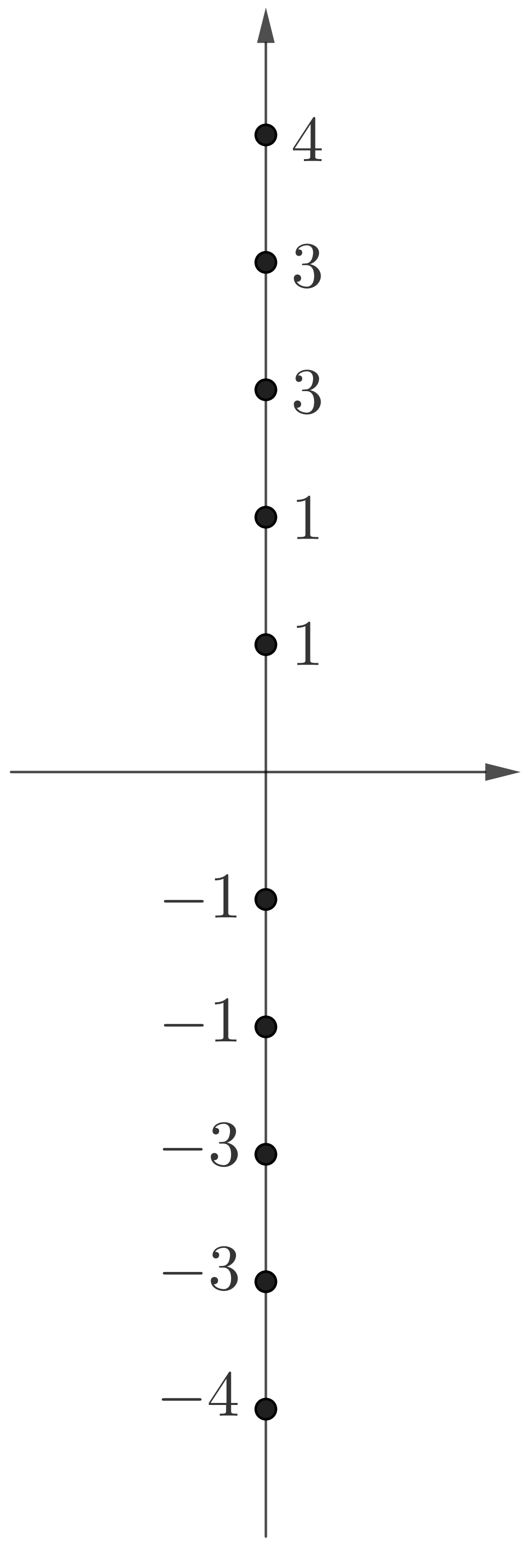
The Stokes constants S_m are obtained analytically from the **local expansions** of $\hat{\phi}(\zeta)$ as

$$\hat{\phi}(\zeta) = -\frac{S_m}{2\pi \mathrm{i}} \log(\zeta - \zeta_m) + \ldots \rightsquigarrow \phi_m(\hbar) = 1 , \quad m \in \mathbb{Z}_{\neq 0} .$$

The first resurgent structure: an analytic approach — II

The **Stokes constants** have the form

$$S_m = S_1 \frac{\alpha_m}{m}, \quad \alpha_{-m} = -\alpha_m, \quad \alpha_m \in \mathbb{N}_+ \quad m > 0, \quad S_1 = 3\sqrt{3}\mathrm{i}.$$



The analytic results are cross-checked numerically.

The exact solution obtained for $\phi(\hbar)$ can be rigorously translated into an exact solution for the original series

$$\psi(\hbar) = \exp(\phi(\hbar))$$

by applying the **Stokes automorphism** formulated in terms of **alien derivatives**.

This analytic solution to the resurgent structure of $\psi(\hbar)$ confirms our previous numerical analysis.

The analytic procedure is then straightforwardly applied to the **dual limit** of $\hbar \to \infty$.

The dual limit

Let now $\psi(\hbar) = \exp(\phi(\hbar))$ be the perturbative expansion for $\hbar \to \infty$.

I perform a **fully analytic resurgent study** as before.

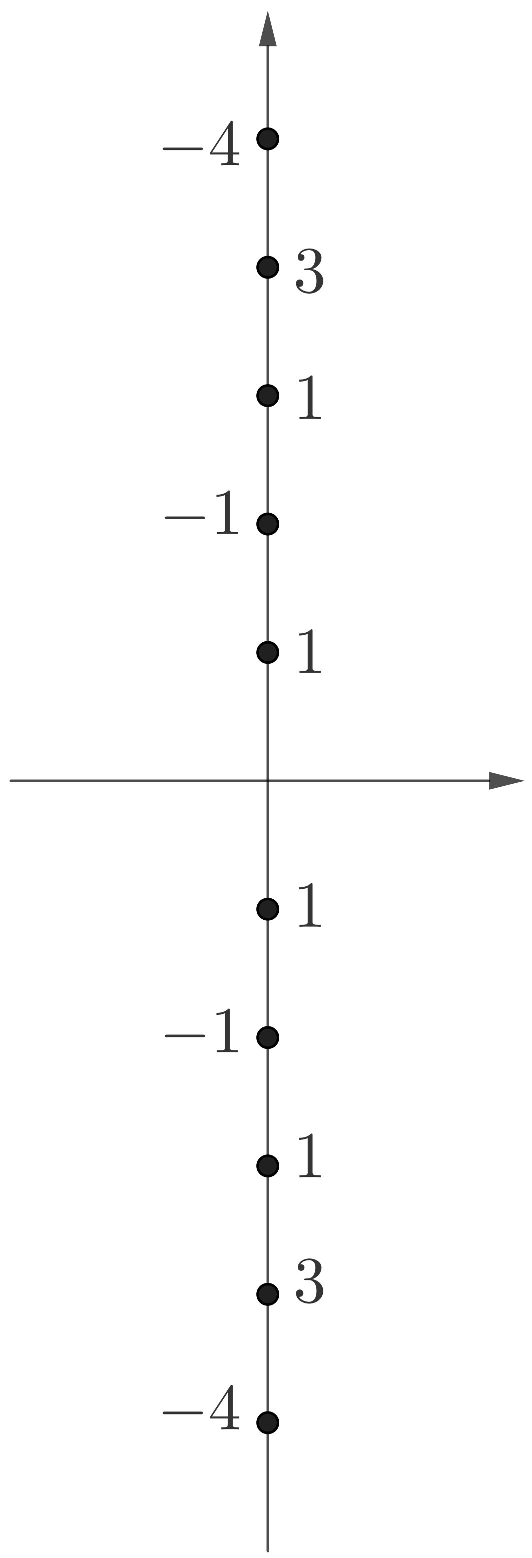
I prove the location and type of the singularities, the local behavior of the Borel transform at the singular points, and the numerical values of the Stokes constants.

I find logarithmic branch points along the imaginary axis at all non-zero integer multiples of $2\pi\mathrm{i}/3$.

The local expansions of $\hat{\phi}(\zeta)$ give trivial non-perturbative sectors, and the Stokes constants are

$$R_m = R_1 \frac{\beta_m}{m}, \quad \beta_{-m} = \beta_m, \quad \beta_m \in \mathbb{Z}_{\neq 0} \quad m > 0, \quad R_1 = 3.$$

My results agree with the previous numerical estimates of [Gu, Mariño, 2022].



Exact formulae for the Stokes constants — I

I find and prove **explicit exact formulae** for the integer numbers α_m, β_m that arise from the Stokes constants S_m, R_m of the asymptotic series $\log Z_{\mathbb{P}^2}(1, \hbar \to 0)$ and $\log Z_{\mathbb{P}^2}(1, \hbar \to \infty)$ as

$$\alpha_m = m\frac{S_m}{S_1} \in \mathbb{Z}_{\neq 0}\,, \quad \beta_m = m\frac{R_m}{R_1} \in \mathbb{Z}_{\neq 0}\,, \quad m \in \mathbb{Z}_{\neq 0}\,.$$

Semiclassical limit $\hbar \to 0$:

$$\alpha_m = \sum_{\substack{d\,|\,m \\ d \equiv_3 1}} \frac{m}{d} - \sum_{\substack{d\,|\,m \\ d \equiv_3 2}} \frac{m}{d}$$

$$\sum_{m>0} \alpha_m x^m = \sum_{k>0} \frac{k x^k}{1 + x^k + x^{2k}}$$

$$\alpha_m = \frac{p_1^{i_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{i_2+1} + (-1)^{i_2}}{p_2 + 1} \cdot p_3^{i_3}$$

Weakly-coupled limit $g_s \to 0$:

$$\beta_m = \sum_{\substack{d\,|\,m \\ d \equiv_3 1}} d - \sum_{\substack{d\,|\,m \\ d \equiv_3 2}} d$$

$$\sum_{m>0} \beta_m x^m = \sum_{k>0} \frac{x^k(1 - x^{2k})}{(1 + x^k + x^{2k})^2}$$

$$\beta_m = \frac{p_1^{i_1+1} - 1}{p_1 - 1} \cdot \frac{(-1)^{i_2} p_2^{i_2+1} + 1}{p_2 + 1}$$

where $m = p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3}$, and $p_i \equiv i \mod 3$, for $i = 1, 2, 3$.

Exact formulae for the Stokes constants — II

I find and prove **explicit generating functions in the form of q-series** for the Stokes constants S_m, R_m of the asymptotic series $\log Z_{\mathbb{P}^2}(1, \hbar \to 0)$ and $\log Z_{\mathbb{P}^2}(1, \hbar \to \infty)$.

In the semiclassical limit $\hbar \to 0$:

$$\sum_{m>0} S_m \tilde{q}^m = -\mathrm{i}\pi - 3\left(\log(w;\,\tilde{q})_\infty - \log(w^{-1};\,\tilde{q})_\infty\right),$$

where $w = e^{2\pi \mathrm{i}/3}$ and $\tilde{q} = e^{-2\pi \mathrm{i}/b^2} = e^{-4\pi^2 \mathrm{i}/3\hbar}$.

In the weakly-coupled limit $g_s \to 0$:

$$\sum_{m>0} R_m q^m = 3\left(\log(q^{2/3};\,q)_\infty - \log(q^{1/3};\,q)_\infty\right),$$

where $q = e^{2\pi \mathrm{i} b^2} = e^{3\mathrm{i}\hbar}$.

Note that $g(q) = \dfrac{(q^{2/3};\,q)_\infty^2}{(q^{1/3};\,q)_\infty}$ and $G(\tilde{q}) = \dfrac{(w;\,\tilde{q})_\infty}{(w^{-1};\,\tilde{q})_\infty^2}$ are the **holomorphic and anti-holomorphic blocks** that appear in the factorization of $Z_{\mathbb{P}^2}(1, \hbar)$.

Exact formulae for the Stokes constants — III

I find and prove **explicit exact formulae in terms of the Riemann zeta functions** for the Stokes constants S_m, R_m of the asymptotic series $\log Z_{\mathbb{P}^2}(1, \hbar \to 0)$ and $\log Z_{\mathbb{P}^2}(1, \hbar \to \infty)$.

In the semiclassical limit $\hbar \to 0$:

$$\sum_{m>0} \frac{S_m}{m^{2k}} = 3\sqrt{3}\mathrm{i} \cdot 3^{-1-2k} \cdot \zeta(2k) \cdot \big(\zeta(2k+1,1/3) - \zeta(2k+1,2/3)\big) \ .$$

In the weakly-coupled limit $g_s \to 0$:

$$\sum_{m>0} \frac{R_m}{m^{2k-1}} = 3\mathrm{i} \cdot 3^{1-2k} \cdot \zeta(2k) \cdot \big(\zeta(2k-1,1/3) - \zeta(2k-1,2/3)\big) \ .$$

These formulae can be regarded as **exact expressions for the perturbative coefficients**, i.e.,

$$b_{2k} = \frac{(-1)^k}{\pi} \frac{\Gamma(2k)}{A^{2k}} \sum_{m>0} \frac{S_m}{m^{2k}} \quad (\hbar \to 0) \, , \quad b_{2k} = \frac{(-1)^k}{\pi} \frac{\Gamma(2k-1)}{A^{2k-1}} \sum_{m>0} \frac{R_m}{m^{2k-1}} \quad (g_s \to 0) \, .$$

CONCLUSIONS

A brief summary and some comments

I described how resurgence can be effectively applied to perturbative expansions arising in the semiclassical regime $\hbar \to 0$ of the spectral theory dual to the topological string theory on a CY threefold.

The resurgent analysis of these expansions unveils a structure of invisible non-perturbative sectors, and associates to them sets of rational Stokes constants.

Building on the work of [Gu, Mariño, 2022], I conjectured in the general case, and proved analytically in the example of local $\mathbb{P}^2$, that peacock patterns occur, and that the Stokes constants have an explicit arithmetic meaning.

The analytic solution for local $\mathbb{P}^2$ in both limits $\hbar \to 0$ and $\hbar \to \infty$ makes symmetries and differences of the two regimes manifest, and hints at an interesting connection to analytic number theory.

A preliminary numerical resurgent analysis of local $\mathbb{F}_0$ in the semiclassical limit is presented in [Rella, 2022].

An important goal for future work is the geometric and physical understanding the non-perturbative sectors unveiled by [Gu, Mariño, 2022 - Rella, 2022], and the identification of our Stokes constants as enumerative invariants of topological strings.

THANK YOU