RESURGENCE, STOKES CONSTANTS, AND ARITHMETIC FUNCTIONS IN TOPOLOGICAL STRING THEORY

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MOTIVATIONS

Enumerative invariants from resurgence

Resurgent asymptotic series arise naturally as perturbative expansions in quantum theories.

The machinery of resurgence uniquely associates them with a non-trivial collection of complex numbers, known as **Stokes constants**, capturing information about the **non-perturbative sectors** of the theory.

In some remarkable cases, the Stokes constants can be (conjecturally) interpreted in terms of **enumerative invariants** based on the counting of BPS states.

- 4d $\mathcal{N} = 2$ supersymmetric gauge theory in the Nekrasov-Shatashvili limit of the Omega-background

[Grassi, Gu, Mariño, 2019]

- Complex Chern-Simons theory on the complement of a hyperbolic knot

[Garoufalidis, Gu, Mariño, 2020]

- Standard topological string theory on a Calabi-Yau threefold for $g_s \to 0$

[Gu, Mariño, 2021]

- Refined topological string theory in the Nekrasov-Shatashvili limit on a Calabi-Yau threefold for $\hbar \to 0$

[Rella, 2022]

FROM TOPOLOGICAL STRINGS TO QUANTUM OPERATORS AND BACK

From topological strings to quantum operators

Let X be a toric Calabi-Yau (CY) threefold.

Local mirror symmetry pairs X with an algebraic curve $\Sigma \subset \mathbb{C}^* \times \mathbb{C}^*$ of genus g_Σ, whose quantization leads to **quantum-mechanical operators**

$$\rho_j \,, \quad j = 1, \ldots, g_\Sigma \,,$$

acting on $L^2(\mathbb{R})$. They are conjectured to be positive-definite and of trace class, under some assumptions on the mass parameters $\vec{\xi}$.
[Grassi, Hatsuda, Mariño, 2014 - Codesido, Grassi, Mariño, 2015]

Their **generalized Fredholm determinant** $\Xi(\vec{\kappa}, \vec{\xi}, \hbar)$ is an entire function of the true complex deformation parameters κ_j.
Its local expansion at $\vec{\kappa} = 0$ is

$$\Xi(\vec{\kappa}, \vec{\xi}, \hbar) = \sum_{N_1 \geq 0} \cdots \sum_{N_{g_\Sigma} \geq 0} \underbrace{Z(\vec{N}, \vec{\xi}, \hbar)}_{\substack{\text{analytic function} \\ \text{of } \hbar \in \mathbb{R}_{>0}}} \kappa_1^{N_1} \cdots \kappa_{g_\Sigma}^{N_{g_\Sigma}} \,,$$

where the coefficient functions $Z(\vec{N}, \vec{\xi}, \hbar)$ are the **fermionic spectral traces**.

From quantum operators to topological strings

The **Topological Strings/Spectral Theory** (TS/ST) correspondence gives

$$Z(\overrightarrow{N}, \vec{\xi}, \hbar) = \frac{1}{(2\pi \mathrm{i})^{g_\Sigma}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu_1 \cdots \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu_{g_\Sigma} \, \mathrm{e}^{J(\vec{\mu}, \vec{\xi}, \hbar) - \overrightarrow{N} \cdot \vec{\mu}} \,,$$

where the chemical potentials μ_j are defined by $\kappa_j = e^{\mu_j}$.
[Hatsuda, Moriyama, Okuyama, 2012 - Grassi, Hatsuda, Mariño, 2014 - Codesido, Grassi, Mariño, 2015]

The **total grand potential** $J(\overrightarrow{\mu}, \vec{\xi}, \hbar)$ can be written as

$$J(\overrightarrow{\mu}, \vec{\xi}, \hbar) = \underbrace{J^{\mathrm{WS}}(\overrightarrow{\mu}, \vec{\xi}, \hbar)}_{\text{worldsheet grand potential}} + \underbrace{J^{\mathrm{WKB}}(\overrightarrow{\mu}, \vec{\xi}, \hbar)}_{\text{WKB grand potential}} \,,$$

where J^{WS} and J^{WKB} encode the contributions from the standard and Nekrasov-Shatashvili (NS) topological strings, respectively.
[Hatsuda, Mariño, Moriyama, Okuyama, 2013]

The string coupling constant g_s is related to the quantum deformation parameter $\hbar$ by

$$g_s = \frac{4\pi^2}{\hbar} \quad (\textbf{strong-weak coupling duality}).$$

THE RESURGENCE TOOLBOX

Let $\phi(z)$ be a **(simple) resurgent Gevrey-1** asymptotic series of the form

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[\![z]\!]\,, \quad a_n \sim A^{-n} n! \quad n \gg 1\,, \quad A \in \mathbb{R}\,.$$

Its **Borel-Laplace resummation** is the two-step process

$$\phi(z) \longrightarrow \underbrace{\hat{\phi}(\zeta) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\zeta^k}_{\substack{\text{locally analytic} \\ \text{at the origin } \zeta = 0 \\ \text{with singularities at} \\ \zeta = \zeta_\omega\,,\, \omega \in \Omega}} \longrightarrow \underbrace{s_\theta(\phi)(z) = \int_{\rho_\theta} \mathrm{e}^{-\zeta}\hat{\phi}(\zeta z)\, d\zeta}_{\substack{\text{locally analytic} \\ \text{in the complex } z\text{-plane} \\ \text{with discontinuities at} \\ \arg(z) = \arg(\zeta_\omega)\,,\, \omega \in \Omega}}\,,$$

where $\rho_\theta = \mathrm{e}^{\mathrm{i}\theta}\,\mathbb{R}_+$, $\theta = \arg(\zeta)$. If ζ_ω is a logarithmic branch point, we have

$$\hat{\phi}(\zeta) = -\frac{S_\omega}{2\pi\mathrm{i}}\log(\zeta - \zeta_\omega) \underbrace{\hat{\phi}_\omega(\zeta - \zeta_\omega)}_{\substack{\text{locally analytic} \\ \text{at } \zeta - \zeta_\omega = 0}} + \ldots\,, \quad S_\omega \in \mathbb{C} \quad (\textbf{Stokes constant}).$$

When $\theta = \arg(\zeta_\omega)$ for some $\omega \in \Omega$, the line ρ_θ is a **Stokes ray**.

The **discontinuity** across the Stokes ray ρ_θ is given by

$$\mathrm{disc}_\theta \phi(z) = s_{\theta_+}(\phi)(z) - s_{\theta_-}(\phi)(z) = \sum_\omega S_\omega \, \mathrm{e}^{-\zeta_\omega/z} s_{\theta_-}(\phi_\omega)(z)\,,$$

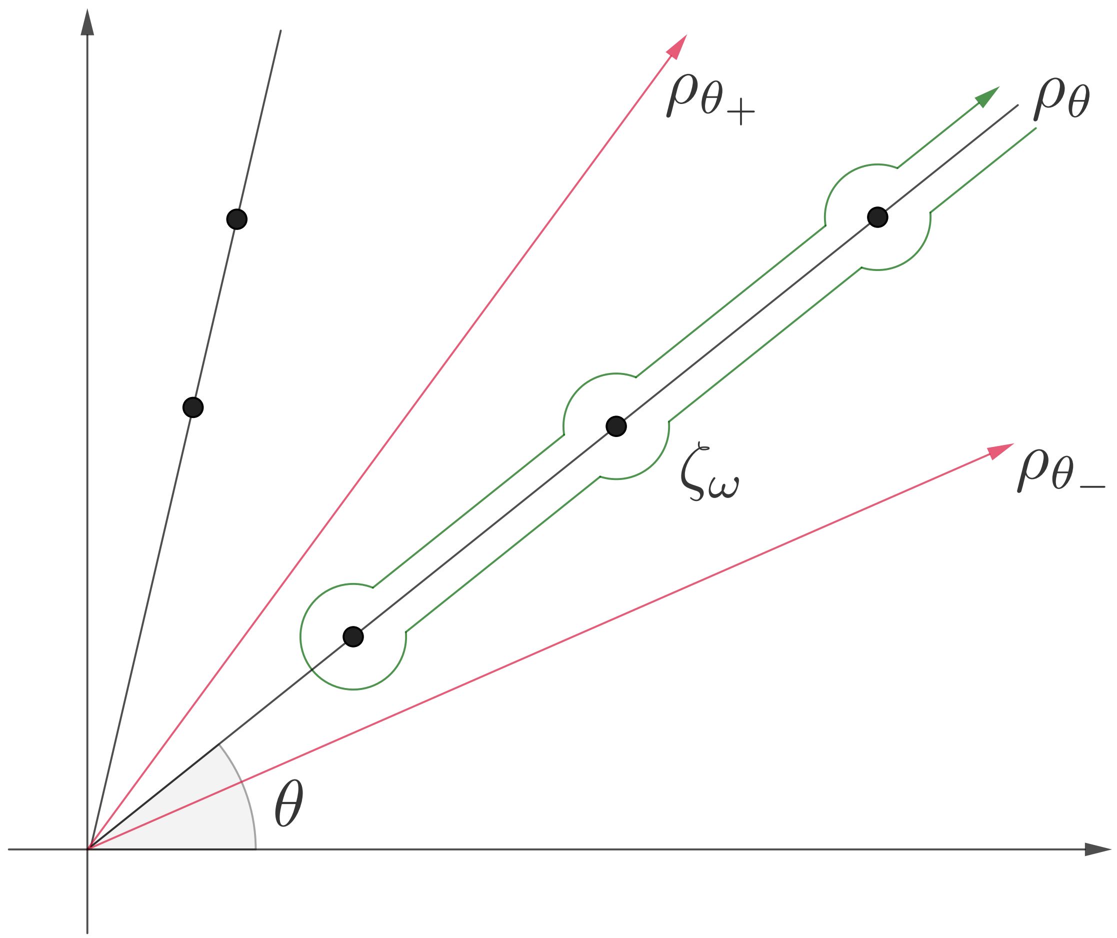
where $\theta_\pm = \theta \pm \epsilon$ for some small positive angle ϵ, and the sum is performed over the singularities ζ_ω with

$$\arg(\zeta_\omega) = \theta\,.$$

The resurgent Gevrey-1 asymptotic series $\phi_\omega(z)$ is the inverse Borel transform of $\hat{\phi}_\omega(\zeta - \zeta_\omega)$.

The **Stokes automorphism** $\mathfrak{S}_\theta$ across ρ_θ is defined by

$$s_{\theta_+} = s_{\theta_-} \circ \mathfrak{S}_\theta\,.$$



Resurgence in quantum theories — III

We can repeat the procedure with each of the series obtained in this way. Schematically,

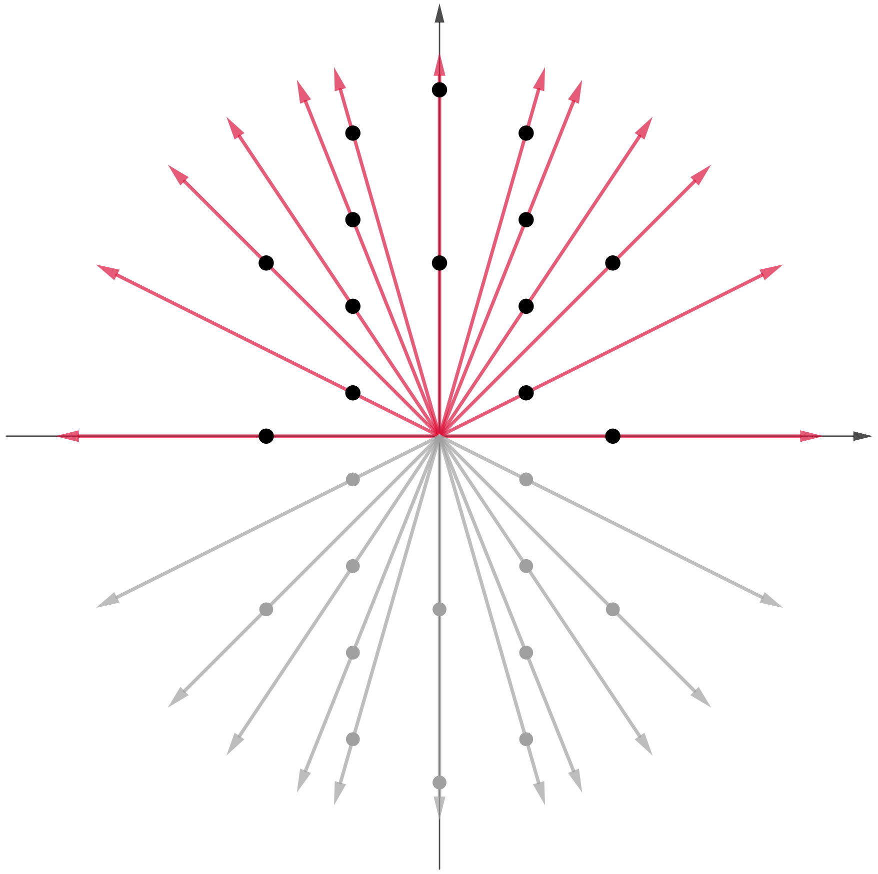
$$\phi \longrightarrow \{\phi_\omega, S_\omega\} \longrightarrow \{\phi_{\omega'}, S_{\omega\omega'}\}\,.$$

Each series in this process can be promoted to **basic trans-series** as

$$\Phi_\omega(z) = \mathrm{e}^{-\zeta_\omega/z}\,\phi_\omega(z)\,.$$

The **minimal resurgent structure** of $\phi(z)$ is

$$\mathfrak{B}_\phi = \underbrace{\left\{\Phi_\omega(z)\right\}_{\omega\in\bar{\Omega}}}_{\substack{\text{smallest subset}\\ \text{closed under } \mathfrak{S}}}\,, \qquad \bar{\Omega}\subseteq\Omega\,.$$



The matrix of Stokes constants is

$$\mathscr{S}_\phi = \{S_{\omega\omega'}\}_{\omega,\omega'\in\bar{\Omega}}\,.$$

Peacock patterns are expected in theories controlled by quantum curves.
[Grassi, Gu, Mariño, 2019 - Garoufalidis, Gu, Mariño, 2020 - 2022 - Gu, Mariño, 2021 - Rella, 2022]

TOPOLOGICAL STRINGS BEYOND PERTURBATION THEORY

Resurgence in topological string theory — I

We assume that $Z(\overrightarrow{N}, \vec{\xi}, \hbar)$ can be analytically continued to $\hbar \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

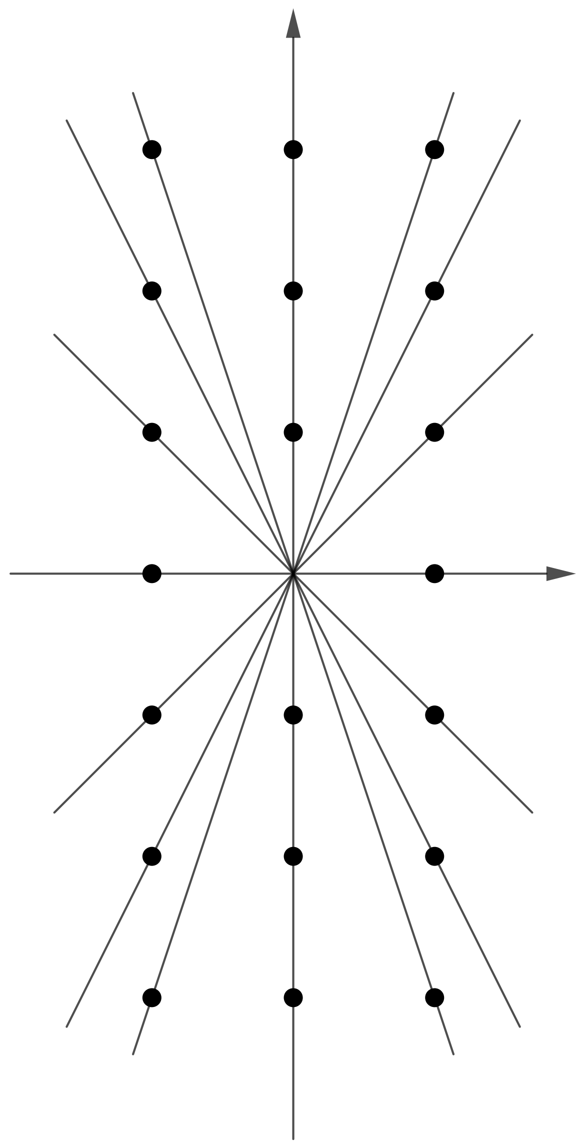
Consider the **semiclassical perturbative expansion**

$$\phi_{\overrightarrow{N}}(\hbar) = \log Z(\overrightarrow{N}, \vec{\xi}, \hbar \to 0)\,, \quad \overrightarrow{N} \in \mathbb{N}^{g_\Sigma}\,,$$

which is a (simple) resurgent Gevrey-1 asymptotic series.

We describe a conjectural proposal for the **minimal resurgent structure** of $\phi_{\overrightarrow{N}}(\hbar)$ at fixed $\overrightarrow{N}$:

$$\underbrace{\Phi_{\sigma,n;\overrightarrow{N}}(\hbar)}_{\substack{\text{infinite family of}\\ \text{basic trans-series}}} = \underbrace{\mathrm{e}^{-n\frac{A}{\hbar}}}_{\substack{\text{non-analytic}\\ \hbar\text{-corrections}}} \underbrace{\phi_{\sigma;\overrightarrow{N}}(\hbar)}_{\substack{\text{Gevrey-1}\\ \text{asymptotic series}}}\,,$$



where $n \in \mathbb{N}$, $\sigma \in \{0, \dots, l\}$, $l \in \mathbb{N}_+$, and $A \in \mathbb{C}$.

The series $\phi_{\sigma;\overrightarrow{N}}(\hbar)$ resurge from $\phi_{\overrightarrow{N}}(\hbar) = \phi_{0;\overrightarrow{N}}(\hbar)$ at the singular points in the Borel plane.

The basic trans-series $\Phi_{\sigma,n;\vec{N}}(\hbar)$ capture **additional, non-perturbative sectors** of the theory.

The corresponding infinitely-many **Stokes constants** are conjectured to be

$$S_{\sigma,\sigma',n;\vec{N}} \in \mathbb{Q}\,, \quad \sigma,\sigma' \in \{0,\ldots,l\}\,, \quad n \in \mathbb{N}\,,$$

They have natural generating functions

$$S_{\sigma,\sigma';\vec{N}}(q) = \sum_{n\in\mathbb{N}} S_{\sigma,\sigma',n;\vec{N}}\, q^n\,,$$

which can be expressed in closed form as q**-series**.

We expect that the Stokes constants $S_{\sigma,\sigma',n;\vec{N}}$ are closely related to non-trivial **enumerative invariants** of the geometry.

In summary,

$$\phi(\hbar) = \log Z(\hbar \to 0) \;\longrightarrow\; \mathfrak{B}_\phi = \{\Phi_{\sigma,n}(\hbar) = \mathrm{e}^{-n\frac{A}{\hbar}}\phi_\sigma(\hbar)\} \;\longrightarrow\; \mathcal{S}_\phi = \{S_{\sigma,\sigma',n} \in \mathbb{Q}\}\,.$$

Resurgence in topological string theory — III

Consider the **dual weakly-coupled regime** $g_s \propto \hbar^{-1} \to 0$.

At fixed $\overrightarrow{N}$, the (simple) resurgent Gevrey-1 asymptotic series

$$\psi_{\overrightarrow{N}}(g_s) = \log Z(\overrightarrow{N}, \vec{\xi}, \hbar \to \infty) \,, \quad \overrightarrow{N} \in \mathbb{N}^{g_\Sigma} \,,$$

is conjectured to have the same resurgent structure described before:

$$\begin{matrix} \text{peacock pattern} \\ \text{in the Borel plane} \end{matrix} \longrightarrow \begin{matrix} \text{infinitely-many} \\ \text{Stokes constants in } \mathbb{Q} \end{matrix} \longrightarrow \begin{matrix} \text{integer invariants} \\ \text{of the geometry} \end{matrix}$$

Some remarks:

1. The asymptotic expansion $Z(\overrightarrow{N}, \vec{\xi}, \hbar \to \infty)$ has an exponential pre-factor of the form e^{-1/g_s} (**conifold volume conjecture for toric CYs**). Its Stokes constants are **integers**.
[Gu, Mariño, 2021]

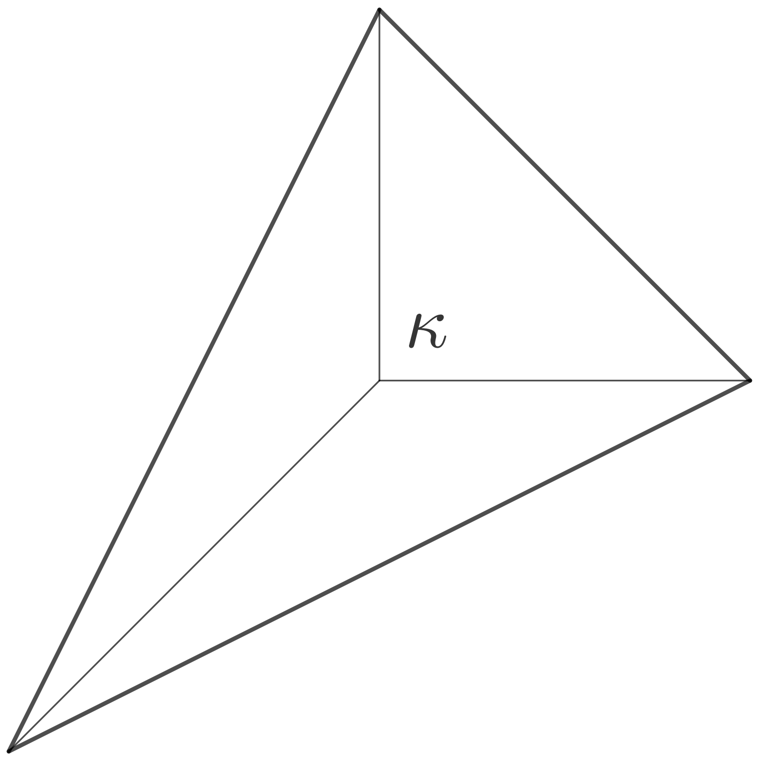
2. The asymptotic expansion $Z(\overrightarrow{N}, \vec{\xi}, \hbar \to 0)$ has no exponential pre-factor of the form $e^{-1/\hbar}$ (**new analytic prediction of the TS/ST correspondence**), and its Stokes constants are generally **complex numbers**.
[Rella, 2022]

LOCAL $\mathbb{P}^2$ — A CASE STUDY

Introduction to the local $\mathbb{P}^2$ geometry

Local $\mathbb{P}^2$ is the total space of the canonical bundle over the projective surface $\mathbb{P}^2$, that is,

$$X = \mathcal{O}(-3) \to \mathbb{P}^2 .$$



It is a **toric del Pezzo CY threefold** with one complex modulus κ and no mass parameters.

The mirror curve is

$$\Sigma \,:\, \mathrm{e}^{x} + \mathrm{e}^{y} + \mathrm{e}^{-x-y} + \kappa = 0 \,, \quad x, y \in \mathbb{C} \,,$$

and its Weyl quantization gives a **quantum-mechanical operator** acting on $L^2(\mathbb{R})$, that is,
[Grassi, Hatsuda, Mariño, 2014]

$$\mathsf{O}_{\mathbb{P}^2}(\mathsf{x}, \mathsf{y}) = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{x}-\mathsf{y}} \,, \quad [\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar \quad \text{(self-adjoint Heisenberg operators).}$$

The inverse operator $\rho_{\mathbb{P}^2} = \mathsf{O}_{\mathbb{P}^2}^{-1}$ is positive-definite and of **trace class**.
[Kashaev, Mariño, 2015]

The fermionic spectral traces $Z_{\mathbb{P}^2}(N, \hbar)$, $N \in \mathbb{N}$, can be expressed as **matrix model integrals**.
[Mariño, Zakany, 2015 - Kashaev, Mariño, Zakany, 2015]

Exact solution to the resurgent structure at weak coupling — I

The first spectral trace is known in **closed form**, showing an explicit factorization into holomorphic/anti-holomorphic blocks.
[Kashaev, Mariño, 2015 - Gu, Mariño, 2021]

We obtain the **all-orders perturbative expansion**

$$Z_{\mathbb{P}^2}(1, \hbar \to 0) = \frac{\Gamma(1/3)^3}{6\pi\hbar} \exp \underbrace{\left(3 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_{2n} B_{2n+1}(2/3)}{2n(2n+1)!} (3\hbar)^{2n} \right)}_{\phi(\hbar) = \sum_{n=1}^{\infty} a_{2n} \hbar^{2n} \ \in \ \mathbb{Q}[[\hbar]]} .$$

We present a **fully analytic solution** to the resurgent structure of $\phi(\hbar)$.
[Rella, 2022]

The coefficients of $\phi(\hbar)$ grow factorially for $n \gg 1$ as

$$a_{2n} \sim (-1)^n (2n)! \left(4\pi^2/3\right)^{-2n} \quad (\textbf{Gevrey-1 asymptotic series}).$$

Proposition: The Borel transform $\hat{\phi}(\zeta)$ can be explicitly resummed into a **well-defined, exact function** of $\zeta \in \mathbb{C}$.

Exact solution to the resurgent structure at weak coupling — II

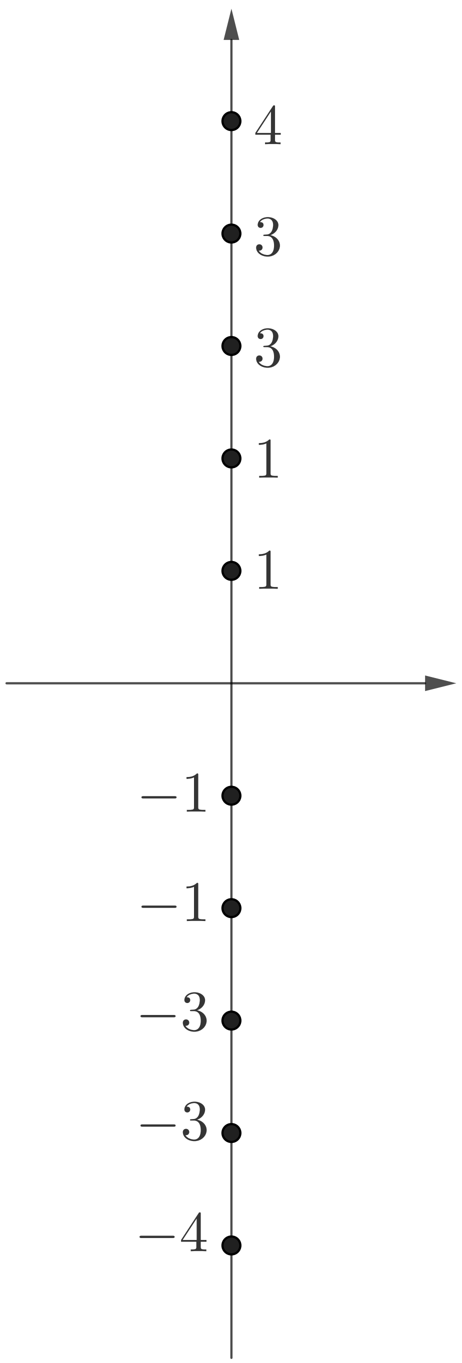
Corollary 1: The Borel transform $\hat{\phi}(\zeta)$ is simple resurgent, and its singularities are **logarithmic branch points** at

$$\zeta_n = \frac{4\pi^2 \mathrm{i}}{3} n\,, \quad n \in \mathbb{Z}_{\neq 0}\,.$$

Corollary 2: The **local expansion** of $\hat{\phi}(\zeta)$ at $\zeta = \zeta_n$ is given by

$$\hat{\phi}(\zeta) = -\frac{S_n}{2\pi \mathrm{i}} \log(\zeta - \zeta_n) + \ldots\,, \quad n \in \mathbb{Z}_{\neq 0}\,,$$

where $\hat{\phi}_n(\zeta) = 1$. The Stokes constants S_n are **accessible analytically**.



Proposition: After being normalized, the Stokes constants S_n are rational numbers and simply related to an **interesting sequence of integers α_n**.

$$S_1 = 3\sqrt{3}\mathrm{i}\,, \quad \frac{S_n}{S_1} = \frac{\alpha_n}{n} \in \mathbb{Q}_{>0} \quad n \in \mathbb{Z}_{\neq 0,1}\,,$$
$$\alpha_n = -\alpha_{-n}\,, \quad \alpha_n \in \mathbb{Z}_{>0} \quad n \in \mathbb{Z}_{>0}\,.$$

Exact solution to the resurgent structure at strong coupling

As before, we obtain an **all-orders perturbative expansion** for $Z_{\mathbb{P}^2}(1, \hbar \to \infty)$, which gives a Gevrey-1 asymptotic series

$$\psi(\tau) = \sum_{n=1}^{\infty} b_{2n} \tau^{2n-1} \in \mathbb{Q}[\pi, \sqrt{3}][\![\tau]\!] , \quad \tau = -\frac{2\pi}{3\hbar} .$$

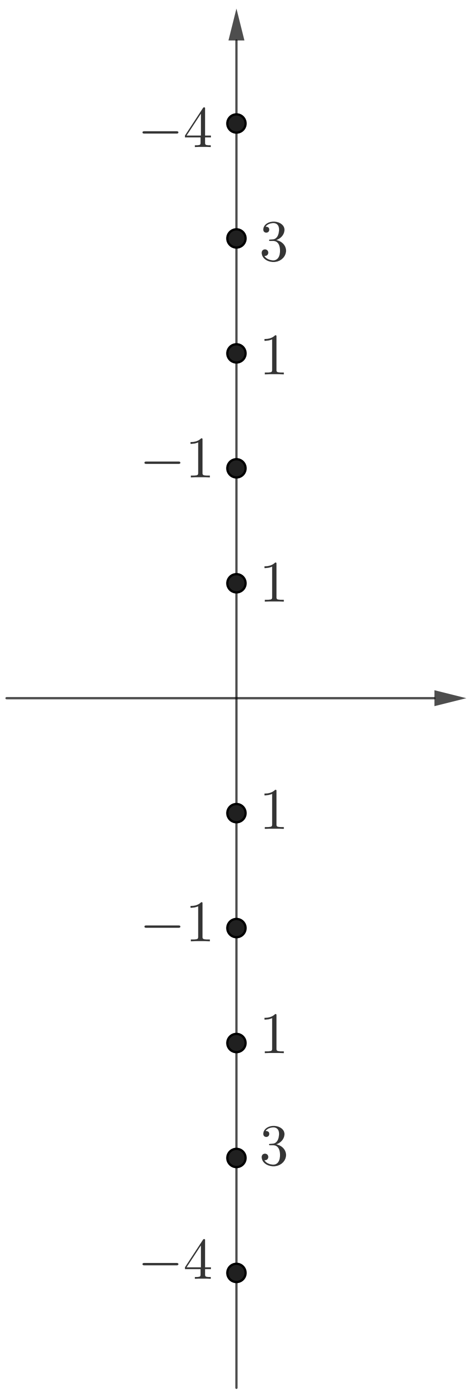
$$b_{2n} \sim (-1)^n (2n)! (2\pi/3)^{-2n} , \quad n \gg 1 .$$

We present a **fully analytic solution** to the resurgent structure of $\psi(\tau)$.
[Rella, 2022]

1. Exact, explicit resummation of $\hat{\psi}(\zeta)$ as a simple resurgent function.
2. Logarithmic branch points at $\zeta_n = \frac{2\pi\mathrm{i}}{3} n, n \in \mathbb{Z}_{\neq 0}$.
3. Local expansion at $\zeta = \zeta_n$:

$$\hat{\psi}(\zeta) = -\frac{R_n}{2\pi\mathrm{i}} \log(\zeta - \zeta_n) + \ldots \longrightarrow \hat{\psi}_n(\zeta) = 1 , \quad n \in \mathbb{Z}_{\neq 0} .$$

$$\boxed{\begin{gathered} R_1 = 3 , \quad R_n = R_1 \frac{\beta_n}{n} \in \mathbb{Q}_{\neq 0} \quad n \in \mathbb{Z}_{\neq 0,1} , \\ \beta_n = \beta_{-n} , \quad \beta_n \in \mathbb{Z}_{\neq 0} \quad n \in \mathbb{Z}_{>0} . \end{gathered}}$$



Closed formulae for the Stokes constants — I

We present **exact number-theoretic statements** on the Stokes constants S_n, R_n, $n \in \mathbb{Z}_{>0}$.
[Rella, 2022]

Proposition: The normalized Stokes constants are **divisor sum functions**.

$$\frac{S_n}{S_1} = \sum_{\substack{d|n \\ d \equiv_3 1}} \frac{1}{d} - \sum_{\substack{d|n \\ d \equiv_3 2}} \frac{1}{d}, \quad \frac{R_n}{R_1} = \sum_{\substack{d|n \\ d \equiv_3 1}} \frac{d}{n} - \sum_{\substack{d|n \\ d \equiv_3 2}} \frac{d}{n}.$$

Corollary 1: The normalized Stokes constants are **multiplicative arithmetic functions**.

$$\frac{S_n}{S_1} = \prod_{p \in \mathbb{P}} \frac{S_{p^e}}{S_1}, \quad \frac{R_n}{R_1} = \prod_{p \in \mathbb{P}} \frac{R_{p^e}}{R_1}, \quad n = \prod_{p \in \mathbb{P}} p^e, \quad e \in \mathbb{N},$$

where S_{p^e} and R_{p^e} are known in closed form.

Corollary 2: The Stokes constants have **generating functions given by q-series**.

$$\sum_{n=1}^{\infty} S_n x^n = -\mathrm{i}\pi - 3\log\frac{(\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}};\, x)_\infty}{(\mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}};\, x)_\infty}, \quad \sum_{n=1}^{\infty} R_n x^{n/3} = 3\log\frac{(x^{2/3};\, x)_\infty}{(x^{1/3};\, x)_\infty}, \quad |x| < 1.$$

Closed formulae for the Stokes constants — II

As a consequence, we obtain **exact expressions for the discontinuities** of $\phi(\hbar), \psi(\tau)$ across the Stokes line on the positive imaginary axis.

$$\mathrm{disc}_{\pi/2}\phi(\hbar) = \sum_{n=1}^{\infty} S_n \mathrm{e}^{-n\frac{4\pi^2}{3}\mathrm{i}\hbar} = -\mathrm{i}\pi - 3\log(\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}};\, \tilde{q})_\infty + 3\log(\mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}};\, \tilde{q})_\infty\,, \quad \tilde{q} = \mathrm{e}^{-\frac{4\pi^2}{3\hbar}\mathrm{i}}\,,$$

$$\mathrm{disc}_{\pi/2}\psi(\tau) = \sum_{n=1}^{\infty} R_n \mathrm{e}^{-n\frac{2\pi}{3}\mathrm{i}\tau} = 3\log(q^{2/3};\, q)_\infty - 3\log(q^{1/3};\, q)_\infty\,, \quad q = \mathrm{e}^{-\frac{2\pi}{\tau}\mathrm{i}} = \mathrm{e}^{3\mathrm{i}\hbar}\,.$$

Proposition: The perturbative coefficients a_{2n}, b_{2n}, $n \in \mathbb{Z}_{>0}$, satisfy the **exact large order relations**

$$a_{2n} = \frac{(-1)^n}{\pi\mathrm{i}}\frac{\Gamma(2n)}{A^{2n}} \underbrace{\sum_{m=1}^{\infty}\frac{S_m}{m^{2n}}}_{\text{L-series}}\,, \quad b_{2n} = \frac{(-1)^n}{\pi}\frac{\Gamma(2n-1)}{A^{2n-1}} \underbrace{\sum_{m=1}^{\infty}\frac{R_m}{m^{2n-1}}}_{\text{L-series}}\,,$$

$$\sum_{m=1}^{\infty}\frac{S_m/S_1}{m^{2n}} = \frac{\zeta(2n)}{3^{2n+1}}\left(\zeta\left(2n+1,\frac{1}{3}\right) - \zeta\left(2n+1,\frac{2}{3}\right)\right)\,,$$

$$\sum_{m=1}^{\infty}\frac{R_m/R_1}{m^{2n-1}} = \frac{\zeta(2n)}{3^{2n-1}}\left(\zeta\left(2n-1,\frac{1}{3}\right) - \zeta\left(2n-1,\frac{2}{3}\right)\right)\,.$$

A bridge to analytic number theory

Recall that the multiplication of Dirichlet series is compatible with the **Dirichlet convolution** of arithmetic functions, that is,

$$f(m) = \big(f_1 * f_2\big)(m)\,,\; m \in \mathbb{Z}_{>0} \;\longrightarrow\; \sum_{m=1}^{\infty} \frac{f(m)}{m^s} = \sum_{m=1}^{\infty} \frac{f_1(m)}{m^s} \sum_{m=1}^{\infty} \frac{f_2(m)}{m^s}\,,\; s \in \mathbb{C}\,,\; \mathfrak{R}(s) > 1\,.$$

Proposition 1: The perturbative coefficients are particular values of a known L-function, which admits a **remarkable factorization dictated by the Dirichlet decomposition** of the Stokes constants.

Proposition 2: The weak and strong coupling L-functions are related by a symmetric **unitary shift** in the arguments of the factors:

$$\frac{S_m}{S_1} = \big(\chi_{3,2}F_{-1} * F_0\big)(m) \;\longrightarrow\; \sum_{m=1}^{\infty} \frac{S_m/S_1}{m^s} = \underbrace{L(s+1,\chi_{3,2})\,\zeta(s)}_{\text{L-function}} \qquad (\hbar \to 0),$$

$$\frac{R_m}{R_1} = \big(\chi_{3,2}F_0 * F_{-1}\big)(m) \;\longrightarrow\; \sum_{m=1}^{\infty} \frac{R_m/R_1}{m^s} = \underbrace{L(s,\chi_{3,2})\,\zeta(s+1)}_{\text{L-function}} \qquad (\hbar \to \infty),$$

where $F_\alpha(m) = m^\alpha$, $\chi_{3,2}(m) = [m]_3$ (non-principal Dirichlet character mod 3).

CONCLUSIONS

Final remarks and open questions

The resurgent analysis of the weak and strong coupling perturbative expansions arising naturally in topological string theory unveils a **universal mathematical structure** of hidden non-perturbative sectors (*peacock patterns*) and **infinitely-many rational Stokes constants** (*enumerative invariants*).

A geometric and physical understanding of the non-perturbative sectors and an explicit identification of the Stokes constants as topological invariants are still missing.

The full resurgent structure of the first spectral trace of local $\mathbb{P}^2$ in both limits is analytically solvable and displays a striking **number-theoretic structure**, which makes the duality between the two regimes manifest.

We would like to test the arithmetic framework for other CY geometries and higher-order spectral traces in support of a potential generalization.

Our asymptotic series can be defined a priori on the topological strings side of the **TS/ST correspondence** directly via the integral representation of the fermionic spectral traces.

A **WKB 't Hooft-like regime** associated to $\hbar \to 0$ is used to present a new analytic prediction on the semiclassical asymptotics of the fermionic spectral traces from the NS topological string in a suitable symplectic frame. Further work is required to obtain a full geometric picture.
[Rella, 2022]

THANK YOU!