RESURGENCE, STOKES CONSTANTS, AND ARITHMETIC FUNCTIONS IN TOPOLOGICAL STRING THEORY

Claudia Rella

Département de Physique Théorique, Université de Genève

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MOTIVATIONS

Enumerative invariants from resurgence

Resurgent asymptotic series arise naturally as perturbative expansions in quantum theories.

The machinery of resurgence uniquely associates them with a non-trivial collection of complex numbers, known as **Stokes constants**, capturing information about the **non-perturbative sectors** of the theory.

In some remarkable cases, the Stokes constants can be (conjecturally) interpreted in terms of **enumerative invariants** based on the counting of BPS states.

- Seiberg–Witten curve of 4d $\mathcal{N} = 2$ super Yang-Mills theory
[Grassi, Gu, Mariño, 2019]
- Complex Chern–Simons theory on Seifert fibered homology spheres
[Andersen, Mistegård, 2018]
- Complex Chern–Simons theory on complements of hyperbolic knots
[Garoufalidis, Gu, Mariño, 2020]
- Standard topological string theory on (toric) Calabi–Yau 3-folds for $g_s \to 0$
[Gu, Mariño, 2021 - 2022 - Rella, 2022 - Gu, Kashani-Poor, Klemm, Mariño, 2023]
- Nekrasov–Shatashvili topological string theory on toric Calabi–Yau 3-folds for $\hbar \to 0$
[Gu, Mariño, 2022 - Rella, 2022]

THE RESURGENCE TOOLBOX

Resurgence in quantum theories — I

Let $\phi(z)$ be a (simple) **resurgent Gevrey-1** asymptotic series of the form

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]]\,, \quad a_n \sim A^{-n} n! \quad n \gg 1\,, \quad A \in \mathbb{R}\,.$$

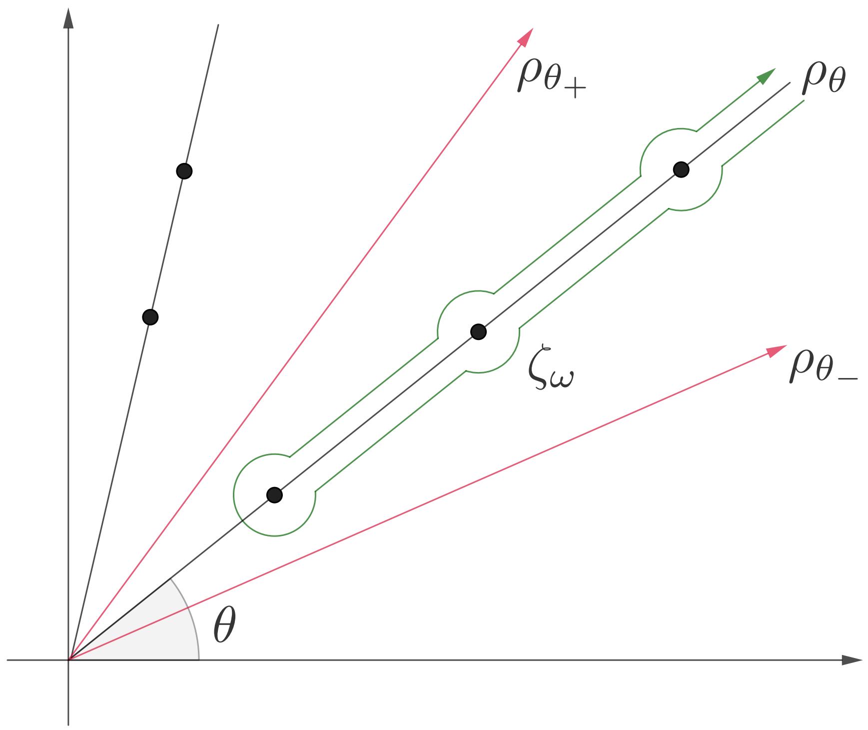
Its **Borel–Laplace resummation** is the two-step process

$$\phi(z) \longrightarrow \underbrace{\hat{\phi}(\zeta) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \zeta^k}_{\substack{\text{locally analytic at } \zeta = 0 \\ \text{with singularities at } \zeta = \zeta_\omega}} \longrightarrow \underbrace{s_\theta(\phi)(z) = \int_{\rho_\theta} \mathrm{e}^{-\zeta} \hat{\phi}(\zeta z)\, d\zeta}_{\substack{\text{locally analytic in the complex } z\text{-plane} \\ \text{with discontinuities at } \arg(z) = \arg(\zeta_\omega)}}\,,$$

where $\rho_\theta = \mathrm{e}^{\mathrm{i}\theta}\,\mathbb{R}_+$, $\theta = \arg(\zeta)$. If ζ_ω is a logarithmic branch point, we have

$$\hat{\phi}(\zeta) = -\frac{S_\omega}{2\pi \mathrm{i}} \log(\zeta - \zeta_\omega) \underbrace{\hat{\phi}_\omega(\zeta - \zeta_\omega)}_{\text{locally analytic at } \zeta - \zeta_\omega = 0} + \ldots\,,$$

where $S_\omega \in \mathbb{C}$ is the **Stokes constant**. When $\theta = \arg(\zeta_\omega)$, the line ρ_θ is a **Stokes ray**.



The **discontinuity** across the Stokes ray ρ_θ is given by

$$\mathrm{disc}_\theta \phi(z) = s_{\theta_+}(\phi)(z) - s_{\theta_-}(\phi)(z) = \\ = \sum_\omega S_\omega \, \mathrm{e}^{-\zeta_\omega/z} \, s_{\theta_-}(\phi_\omega)(z) \, .$$

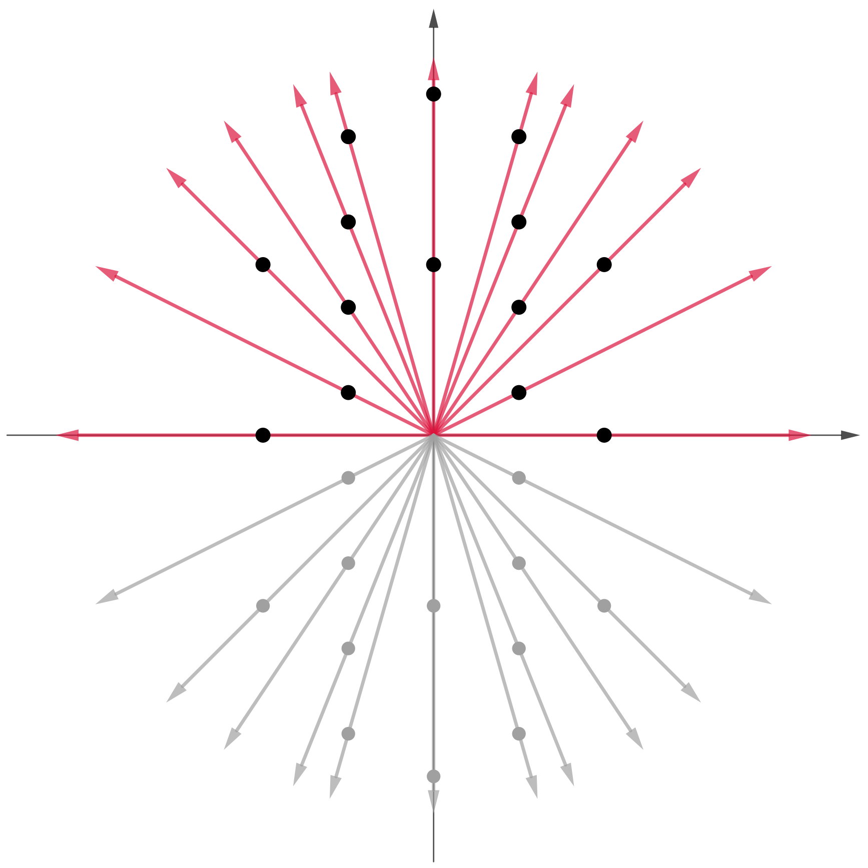
The **Stokes automorphism** $\mathfrak{S}_\theta$ across ρ_θ is defined by

$$s_{\theta_+} = s_{\theta_-} \circ \mathfrak{S}_\theta \, .$$

Schematically, $\phi \longrightarrow \{\phi_\omega, S_\omega\} \longrightarrow \{\phi_{\omega'}, S_{\omega\omega'}\}$.
Each series can be promoted to **basic trans-series** as

$$\Phi_\omega(z) = \mathrm{e}^{-\zeta_\omega/z} \, \phi_\omega(z) \, .$$

The **minimal resurgent structure** of $\phi(z)$ is the smallest subset of $\left\{\Phi_\omega(z)\right\}$ which is closed under $\mathfrak{S}$.



Peacock patterns are expected in theories controlled by quantum curves.

TOPOLOGICAL STRINGS BEYOND PERTURBATION THEORY

From topological strings to quantum operators and back

Let X be a toric Calabi–Yau 3-fold. The Weyl quantization of its mirror curve $\Sigma \subset \mathbb{C}^* \times \mathbb{C}^*$ leads to (positive-definite and trace-class) **quantum operators** $\rho_{j=1,\ldots,g_\Sigma}$, acting on $L^2(\mathbb{R})$.
[Grassi, Hatsuda, Mariño, 2014 - Codesido, Grassi, Mariño, 2015]

Their generalized Fredholm determinant defines the **fermionic spectral traces** by

$$\Xi(\vec{\kappa}, \vec{\xi}, \hbar) = \sum_{N_1 \geq 0} \cdots \sum_{N_{g_\Sigma} \geq 0} Z(\vec{N}, \vec{\xi}, \hbar)\, \kappa_1^{N_1} \cdots \kappa_{g_\Sigma}^{N_{g_\Sigma}} .$$

The **total grand potential** of topological string theory on X can be written as
[Hatsuda, Mariño, Moriyama, Okuyama, 2013]

$$J(\vec{\mu}, \vec{\xi}, \hbar) = \underbrace{J^{\mathrm{WS}}(\vec{\mu}, \vec{\xi}, \hbar)}_{\substack{\text{standard} \\ \text{topological string}}} + \underbrace{J^{\mathrm{WKB}}(\vec{\mu}, \vec{\xi}, \hbar)}_{\substack{\text{Nekrasov-Shatashvili} \\ \text{topological string}}} .$$

The **Topological Strings/Spectral Theory** correspondence states that
[Hatsuda, Moriyama, Okuyama, 2012 - Grassi, Hatsuda, Mariño, 2014 - Codesido, Grassi, Mariño, 2015]

$$Z(\vec{N}, \vec{\xi}, \hbar) = \frac{1}{(2\pi \mathrm{i})^{g_\Sigma}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu_1 \cdots \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu_{g_\Sigma}\, \mathrm{e}^{J(\vec{\mu}, \vec{\xi}, \hbar) - \vec{N} \cdot \vec{\mu}} , \quad (\kappa_j = \mathrm{e}^{\mu_j}) .$$

Let $Z(\overrightarrow{N}, \vec{\xi}, \hbar)$ be analytically continued to $\hbar \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Consider the **semiclassical perturbative expansion**

$$\phi_{\overrightarrow{N}}(\hbar) = \log Z(\overrightarrow{N}, \vec{\xi}, \hbar \to 0)\,, \quad \overrightarrow{N} \in \mathbb{N}^{g_\Sigma}\,,$$

which is a (simple) resurgent Gevrey-1 asymptotic series.

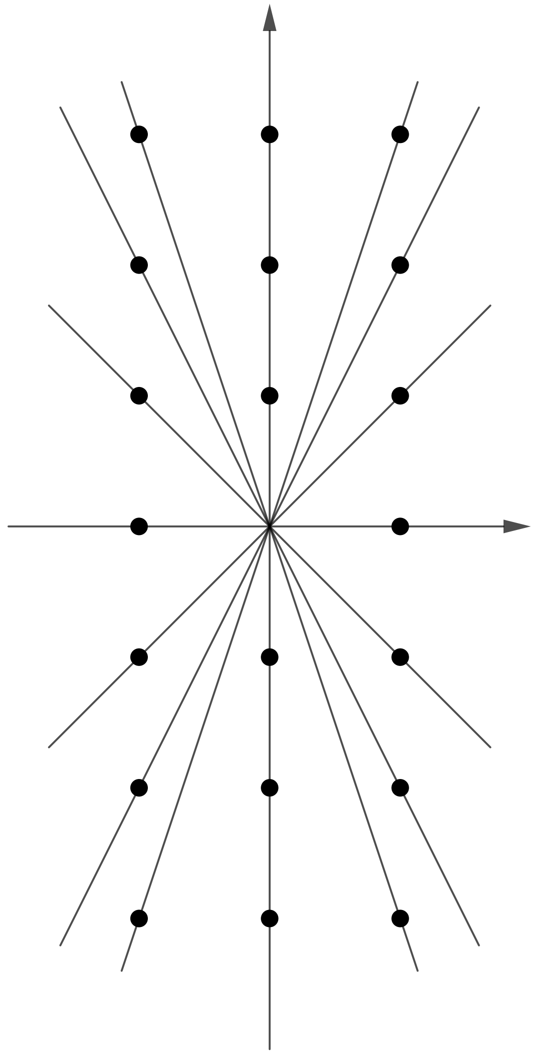
We describe a conjectural proposal for the **minimal resurgent structure** of $\phi_{\overrightarrow{N}}(\hbar)$ at fixed $\overrightarrow{N}$:

$$\Phi_{\sigma,n;\overrightarrow{N}}(\hbar) = \mathrm{e}^{-n\frac{A}{\hbar}}\,\phi_{\sigma;\overrightarrow{N}}(\hbar) \quad \text{(peacock patterns)},$$

where $n \in \mathbb{N}$, $\sigma \in \{0, \ldots, l\}$, $l \in \mathbb{N}_+$, and $A \in \mathbb{C}$.

The corresponding **Stokes constants** satisfy

$$S_{\sigma,\sigma',n;\overrightarrow{N}} \in \mathbb{Q}\,, \quad S_{\sigma,\sigma';\overrightarrow{N}}(q) = \sum_{n \in \mathbb{N}} S_{\sigma,\sigma',n;\overrightarrow{N}}\, q^n \quad \text{(q-series)},$$



and they give non-trivial **enumerative invariants** of the geometry.

Resurgence in topological string theory — II

Consider the **dual weakly-coupled regime** $g_s \propto \hbar^{-1} \to 0$ (strong-weak coupling duality).

At fixed $\overrightarrow{N}$, the (simple) resurgent Gevrey-1 asymptotic series

$$\psi_{\overrightarrow{N}}(g_s) = \log Z(\overrightarrow{N}, \vec{\xi}, \hbar \to \infty)\,, \quad \overrightarrow{N} \in \mathbb{N}^{g_\Sigma}\,,$$

is conjectured to have <u>the same resurgent structure</u> described before:

peacock pattern in the Borel plane	$\longrightarrow$	infinitely many Stokes constants in $\mathbb{Q}$	$\longrightarrow$	integer invariants of the geometry

Some remarks:

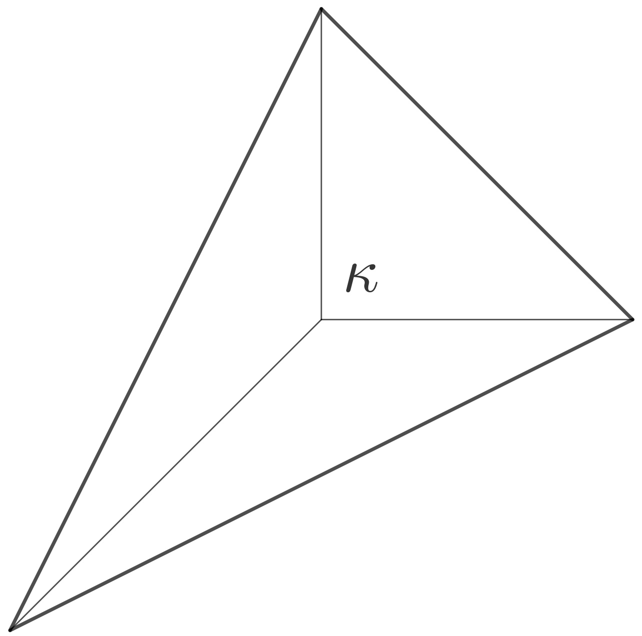
1. The asymptotic expansion $Z(\overrightarrow{N}, \vec{\xi}, \hbar \to \infty)$ has an exponential pre-factor of the form e^{-1/g_s} (**conifold volume conjecture for toric CYs**).
[Gu, Mariño, 2021]

2. The asymptotic expansion $Z(\overrightarrow{N}, \vec{\xi}, \hbar \to 0)$ has no exponential pre-factor of the form $e^{-1/\hbar}$ (**new analytic prediction of the TS/ST correspondence**).
[Rella, 2022]

LOCAL $\mathbb{P}^2$ — A CASE STUDY

Exact solution to the resurgent structure at weak coupling — I

Local $\mathbb{P}^2$ is the total space of the canonical bundle over the projective surface $\mathbb{P}^2$. It is a **toric del Pezzo CY 3-fold** with one complex modulus κ and no mass parameters.



The first spectral trace $Z_{\mathbb{P}^2}(1,\hbar)$ is known in **closed form**, showing an explicit factorization into holomorphic/anti-holomorphic blocks.
[Kashaev, Mariño, 2015 - Gu, Mariño, 2021]

We obtain an **all-orders perturbative expansion** for $Z_{\mathbb{P}^2}(1, \hbar \to 0)$, which gives a Gevrey-1 asymptotic series

$$\phi(\hbar) = \sum_{n=1}^{\infty} a_{2n}\hbar^{2n} \in \mathbb{Q}[[\hbar]] , \quad a_{2n} \sim (-1)^n (2n)! \left(4\pi^2/3\right)^{-2n} \quad n \gg 1 .$$

We present a **fully analytic solution** to the resurgent structure of $\phi(\hbar)$.
[Rella, 2022]

<u>*Proposition:*</u> The Borel transform $\hat{\phi}(\zeta)$ can be explicitly resummed into a **well-defined, exact function** of $\zeta \in \mathbb{C}$.

Exact solution to the resurgent structure at weak coupling — II

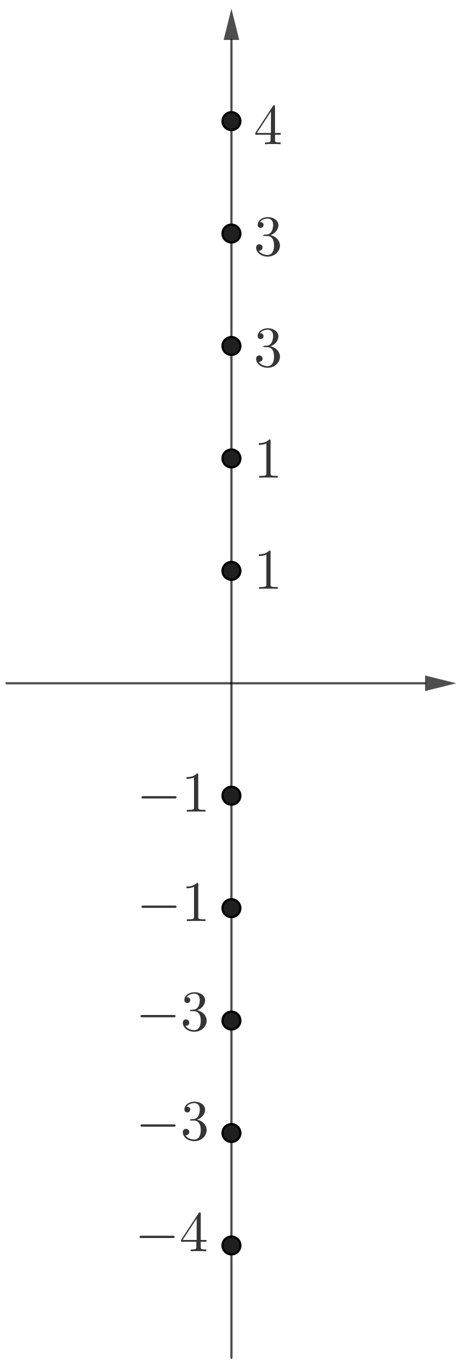
<u>*Corollary 1:*</u> The Borel transform $\hat{\phi}(\zeta)$ is simple resurgent, and its singularities are **logarithmic branch points** at

$$\zeta_n = \frac{4\pi^2 \mathrm{i}}{3} n \,, \quad n \in \mathbb{Z}_{\neq 0} \,.$$

<u>*Corollary 2:*</u> The **local expansion** of $\hat{\phi}(\zeta)$ at $\zeta = \zeta_n$ is given by

$$\hat{\phi}(\zeta) = -\frac{S_n}{2\pi \mathrm{i}} \log(\zeta - \zeta_n) + \dots \,, \quad n \in \mathbb{Z}_{\neq 0} \,,$$

where $\hat{\phi}_n(\zeta) = 1$. The Stokes constants S_n are **accessible analytically**.



<u>*Proposition:*</u> After being normalized, the Stokes constants S_n are rational numbers and simply related to a **non-trivial sequence of integers α_n**.

$$S_1 = 3\sqrt{3}\mathrm{i} \,, \quad \frac{S_n}{S_1} = \frac{\alpha_n}{n} \in \mathbb{Q}_{>0} \quad n \in \mathbb{Z}_{\neq 0,1} \,,$$
$$\alpha_n = -\alpha_{-n} \,, \quad \alpha_n \in \mathbb{Z}_{>0} \quad n \in \mathbb{Z}_{>0} \,.$$

Exact solution to the resurgent structure at strong coupling

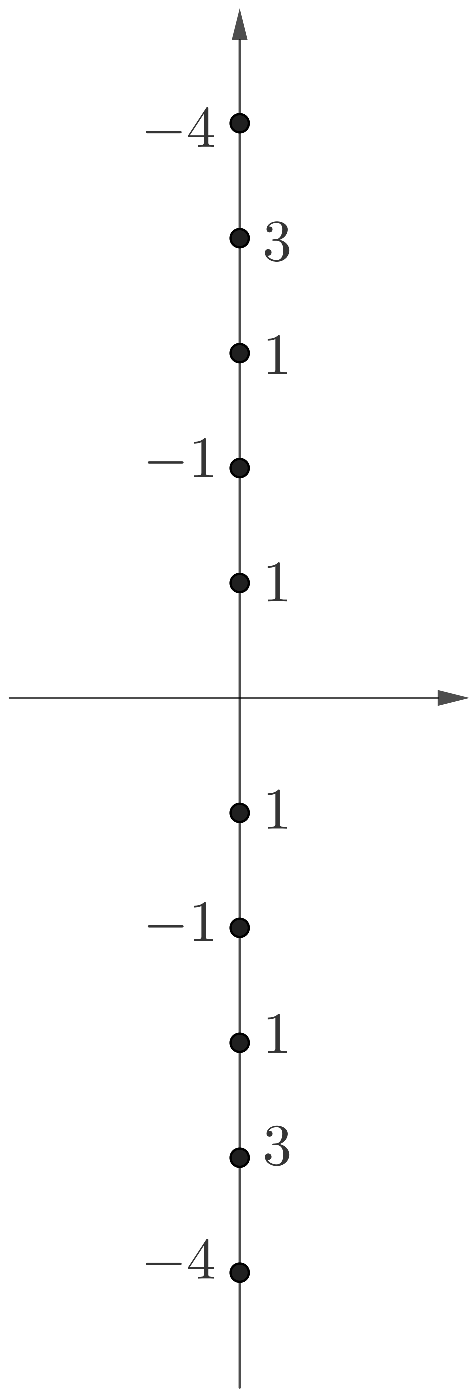
Analogously, we obtain an **all-orders perturbative expansion** for $Z_{\mathbb{P}^2}(1, \hbar \to \infty)$, which gives a Gevrey-1 asymptotic series

$$\psi(\tau) = \sum_{n=1}^{\infty} b_{2n} \tau^{2n-1} \in \mathbb{Q}[\pi, \sqrt{3}][[\tau]] , \quad \tau = -\frac{2\pi}{3\hbar} .$$
$$b_{2n} \sim (-1)^n (2n)! (2\pi/3)^{-2n} , \quad n \gg 1 .$$

We present a **fully analytic solution** to the resurgent structure of $\psi(\tau)$: [Rella, 2022]

1. Exact, explicit resummation of $\hat{\psi}(\zeta)$ as a simple resurgent function.
2. Logarithmic branch points at $\zeta_n = \frac{2\pi i}{3} n, n \in \mathbb{Z}_{\neq 0}$.
3. Local expansion at $\zeta = \zeta_n$:

$$\hat{\psi}(\zeta) = -\frac{R_n}{2\pi i} \log(\zeta - \zeta_n) + \ldots \longrightarrow \hat{\psi}_n(\zeta) = 1 , \quad n \in \mathbb{Z}_{\neq 0} .$$



Proposition: $R_1 = 3 , \quad R_n = R_1 \frac{\beta_n}{n} \in \mathbb{Q}_{\neq 0} \quad n \in \mathbb{Z}_{\neq 0,1} ,$
$\beta_n = \beta_{-n} , \quad \beta_n \in \mathbb{Z}_{\neq 0} \quad n \in \mathbb{Z}_{>0} .$

Closed formulae for the Stokes constants

We present **exact number-theoretic statements** on the Stokes constants $S_n, R_n, n \in \mathbb{Z}_{>0}$.
[Rella, 2022]

Proposition 1: The normalized Stokes constants are **divisor sum functions**:

$$\frac{S_n}{S_1} = \sum_{\substack{d|n \\ d \equiv_3 1}} \frac{1}{d} - \sum_{\substack{d|n \\ d \equiv_3 2}} \frac{1}{d}, \quad \frac{R_n}{R_1} = \sum_{\substack{d|n \\ d \equiv_3 1}} \frac{d}{n} - \sum_{\substack{d|n \\ d \equiv_3 2}} \frac{d}{n}.$$

Proposition 2: The Stokes constants have **generating functions given by q-series**. As a consequence, we obtain **exact expressions for the discontinuities** of $\phi(\hbar), \psi(\tau)$ across the positive imaginary axis:

$$\mathrm{disc}_{\frac{\pi}{2}} \phi(\hbar) = \sum_{n=1}^{\infty} S_n \tilde{q}^n = -\mathrm{i}\pi - 3 \log \frac{(w; \tilde{q})_\infty}{(w^{-1}; \tilde{q})_\infty},$$
$$\mathrm{disc}_{\frac{\pi}{2}} \psi(\tau) = \sum_{n=1}^{\infty} R_n q^n = 3 \log \frac{(q^{2/3}; q)_\infty}{(q^{1/3}; q)_\infty},$$

where $w = \mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}$, $\tilde{q} = \mathrm{e}^{2\pi\mathrm{i}\tau} = \mathrm{e}^{-\frac{4\pi^2}{3\hbar}\mathrm{i}}$, and $q = \mathrm{e}^{-\frac{2\pi}{\tau}\mathrm{i}} = \mathrm{e}^{3\mathrm{i}\hbar}$.

A bridge to analytic number theory — I

Proposition 1: The normalized Stokes constants are **multiplicative arithmetic functions**:

$$\frac{S_n}{S_1} = \prod_{p \in \mathbb{P}} \frac{S_{p^e}}{S_1}, \quad \frac{R_n}{R_1} = \prod_{p \in \mathbb{P}} \frac{R_{p^e}}{R_1}, \quad n = \prod_{p \in \mathbb{P}} p^e, \quad e \in \mathbb{N},$$

where S_{p^e} and R_{p^e} are known in closed form.

Proposition 2: The perturbative coefficients $a_{2n}, b_{2n}, n \in \mathbb{Z}_{>0}$, satisfy **exact large-order relations**:

$$a_{2n} = \frac{(-1)^n}{\pi \mathrm{i}} \frac{\Gamma(2n)}{A^{2n}} \underbrace{\sum_{m=1}^{\infty} \frac{S_m}{m^{2n}}}_{\substack{\text{Dirichlet series} \\ \text{evaluated at } 2n}}, \quad b_{2n} = \frac{(-1)^n}{\pi} \frac{\Gamma(2n-1)}{A^{2n-1}} \underbrace{\sum_{m=1}^{\infty} \frac{R_m}{m^{2n-1}}}_{\substack{\text{Dirichlet series} \\ \text{evaluated at } 2n-1}}.$$

Corollary 1: The Dirichlet series defined by the Stokes constants satisfy an **Euler product expansion** indexed by the set of prime numbers. They are, indeed, L-series.

Corollary 2: The perturbative coefficients $a_{2n}, b_{2n}, n \in \mathbb{Z}_{>0}$, are values of **L-series evaluated at integer points**.

A bridge to analytic number theory — II

Recall that the multiplication of Dirichlet series is compatible with the **Dirichlet convolution** of arithmetic functions, that is,

$$f(m) = \big(f_1 * f_2\big)(m)\,,\; m \in \mathbb{Z}_{>0} \;\longrightarrow\; \sum_{m=1}^{\infty} \frac{f(m)}{m^s} = \sum_{m=1}^{\infty} \frac{f_1(m)}{m^s} \sum_{m=1}^{\infty} \frac{f_2(m)}{m^s}\,,\; s \in \mathbb{C}\,,\; \Re(s) > 1\,.$$

Theorem: The weak and strong coupling L-series **factorise according to the Dirichlet decomposition** of the Stokes constants into the product of two well-known L-functions:

$$\frac{S_m}{S_1} = \big(\chi_{3,2} F_{-1} * F_0\big)(m) \;\longrightarrow\; \sum_{m=1}^{\infty} \frac{S_m/S_1}{m^s} = L(s+1, \chi_{3,2})\,\zeta(s) \qquad (\hbar \to 0),$$

$$\frac{R_m}{R_1} = \big(\chi_{3,2} F_0 * F_{-1}\big)(m) \;\longrightarrow\; \sum_{m=1}^{\infty} \frac{R_m/R_1}{m^s} = L(s, \chi_{3,2})\,\zeta(s+1) \qquad (\hbar \to \infty),$$

where $F_\alpha(m) = m^\alpha$, $\chi_{3,2}(m) = [m]_3$.

Corollary: The weak and strong coupling L-functions are related by a symmetric **unitary shift** in the arguments of the factors.

CONCLUSIONS

Final remarks and work in progress

The resurgent analysis of the perturbative expansions of the topological string on a toric CY 3-fold unveils a **universal mathematical structure** of non-perturbative sectors (*peacock patterns*) and Stokes constants (*enumerative invariants*), whose geometric and physical understanding is still missing.

The example of the first spectral trace of local $\mathbb{P}^2$ displays a striking **number-theoretic fabric**, which we would like to test in other CY geometries and higher-order spectral traces.

A preliminary numerical analysis of the first spectral trace of local $\mathbb{F}_0$ for $\hbar \to 0$ shows logarithmic-type terms in the sub-leading asymptotics.
[Rella, 2022]

Moreover, our exact results on the first spectral trace of local $\mathbb{P}^2$ fit within a broader research program investigating the connection between the **resurgent properties of q-series and quantum modular forms**. Work in progress with V. Fantini and M. Kontsevich.

Finally, our asymptotic series can be defined a priori on the topological strings side of the TS/ST correspondence directly.

A **WKB 't Hooft-like regime** associated to $\hbar \to 0$ provides a new analytic prediction on the semiclassical asymptotics of the fermionic spectral traces from the NS topological string in a suitable symplectic frame. Further work is required to obtain a full geometric picture.
[Rella, 2022]

THANK YOU!